1. Which of the following sets of natural numbers is inductive? (In each case, say whether the set $S$ is inductive or not, and explain why.)

(a) $S = \{x \in \mathbb{N} : x^2 \leq 1000\}$.
(b) $S = \{x \in \mathbb{N} : x^2 \geq 1000\}$.
(c) $S = \emptyset$.
(d) $S = \{x \in \mathbb{N} : (\exists k \in \mathbb{N})x = 2k\}$.
(e) $S = \{x \in \mathbb{N} : x^2 \geq 1000 \lor x = 3\}$.

2. Which of the following statements is necessarily true if $S$ is an inductive set of natural numbers?

(a) $6 \in S \land 11 \in S$.
(b) $6 \in S \lor 11 \in S$.
(c) $6 \in S \implies 11 \in S$.
(d) If $n \in S$ then $n + 1 \in S$.
(e) If $n + 1 \in S$ then $n \in S$.
(f) If $n \in S$ then $n + 3000 \in S$.
(g) $S \neq \emptyset$.

3. Prove each of the following statements. (In each case, first verify that the statement is true for $n = 1, 2, 3$.)

(a) If $n$ is an arbitrary natural number then $8^n - 1$ is divisible by 7.
(b) If $n$ is an arbitrary natural number then

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n + 1)^2}{4}.$$
(c) If \( n \) is an arbitrary natural number then for every positive real number \( x \)
\[(1 + x)^n \geq 1 + nx.
\]

(d) If \( n \) is an arbitrary natural number then
\[
\prod_{k=1}^{n}(2k - 1) = \frac{(2n)!}{2^n n!}.
\]