Prize offered
for the solution of
a problem on tangent cones

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I am offering a u$500 prize to the first person who proves or disproves the following statement:

Let $n$ be a positive integer, and let $S_1, S_2$ be closed subsets of $\mathbb{R}^n$ such that $0 \in S_1 \cap S_2$. Let $C_1, C_2$ be convex cones in $\mathbb{R}^n$ such that

- $C_1$ is a Boltyanskii approximating cone to $S_1$ at $0$;
- $C_2$ is the Clarke tangent cone of $S_2$ at $0$;
- $C_1$ and $C_2$ are strongly transversal.

Then $0$ belongs to the closure of $(S_1 \cap S_2) \setminus \{0\}$. That is, the set $S_1 \cap S_2$ contains points other than $0$ arbitrarily close to $0$.

If the problem is solved simultaneously by more than one person, then the prize money will be split among the authors of the first solution or solutions.

The relevant definitions are as follows:

DEFINITION 1. A cone in a real linear space $X$ is a nonempty subset $C$ of $X$ such that

- whenever $c \in C$, $r \in \mathbb{R}$, and $r \geq 0$, it follows that $rc \in C$.

(Notice that if $C$ is a cone then $0 \in C$.)

DEFINITION 2. Let $S$ be a subset of $\mathbb{R}^n$, and let $s_* \in S$. A Boltyanskii approximating cone to $S$ at $s_*$ is a convex cone $C$ in $\mathbb{R}^n$ such that

- there exist a positive integer $m$, a convex cone $D$ in $\mathbb{R}^m$, an open subset $U$ of $\mathbb{R}^m$ containing $0$, a continuous map $\varphi : U \cap D \mapsto S$, and a linear map $L : \mathbb{R}^m \mapsto \mathbb{R}^n$, such that

$$
\varphi(v) = s_* + L \cdot v + o(||v||) \quad \text{as} \quad v \to_D 0
$$

and

$$
LD = C.
$$
DEFINITION 3. Let $S$ be a closed subset of $\mathbb{R}^n$, and let $s_* \in S$. The Bouligand tangent cone of $S$ at $s_*$ is the set of all $v \in \mathbb{R}^n$ such that

- there exist a sequence $(s_k)_{k \in \mathbb{N}}$ of members of $S$ and a sequence $(h_k)_{k \in \mathbb{N}}$ of strictly positive real numbers such that $v = \lim_{k \to \infty} \frac{s_k - s_*}{h_k}$. \hfill \Box$

From now on, $T_s^{\text{Boul}} S$ denotes the Bouligand tangent cone of $S$ at $s$.

DEFINITION 4. Let $S$ be a closed subset of $\mathbb{R}^n$, and let $s_* \in S$. The Clarke tangent cone of $S$ at $s_*$ is the set of all $v \in \mathbb{R}^n$ such that

- for every sequence $(s_k)_{k \in \mathbb{N}}$ of members of $S$ such that $\lim_{k \to \infty} s_k = s_*$, there exist vectors $v_k \in T_{s_k}^{\text{Boul}} S$ such that $v = \lim_{k \to \infty} v_k$. \hfill \Box$

From now on, $T_s^C S$ denotes the Clarke tangent cone of $S$ at $s$.

DEFINITION 5. Let $C_1$, $C_2$ be convex cones in $\mathbb{R}^n$. We say that $C_1$ and $C_2$ are transversal if $C_1 - C_2 = \mathbb{R}^n$. (Here I am using $C_1 - C_2$ to denote the set $\{c_1 - c_2 : c_1 \in C_1, c_2 \in C_2\}$.)$ \hfill \Box$

REMARK. The convex cones $C_1$ and $C_2$ are transversal if and only if they are not linearly separated. (We say that $C_1$ and $C_2$ are linearly separated if there exists a nontrivial linear functional $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that $\lambda(c) \geq 0$ for all $c \in C_1$ and $\lambda(c) \leq 0$ for all $c \in C_2$.)

DEFINITION 6. Let $C_1$, $C_2$ be convex cones in $\mathbb{R}^n$. We say that $C_1$ and $C_2$ are strongly transversal if

- $C_1$ and $C_2$ are transversal;
- $C_1 \cap C_2 \neq \{0\}$ (that is, $C_1 \cap C_2$ contains a half line through the origin). \hfill \Box

**MOTIVATION**

The question is motivated by the fact that the following two theorems are true:

THEOREM 1. Let $n$ be a positive integer, and let $S_1$, $S_2$ be subsets of $\mathbb{R}^n$ such that $0 \in S_1 \cap S_2$. Let $C_1$, $C_2$ be convex cones in $\mathbb{R}^n$ such that

- $C_1$ is a Boltyanski approximating cone to $S_1$ at 0;
- $C_2$ is a Boltyanski approximating cone to $S_2$ at 0;
- $C_1$ and $C_2$ are strongly transversal.

Then 0 belongs to the closure of $(S_1 \cap S_2) \setminus \{0\}$.

THEOREM 2. Let $n$ be a positive integer, and let $S_1$, $S_2$ be closed subsets of $\mathbb{R}^n$ such that $0 \in S_1 \cap S_2$. Assume that the Clarke tangent cones $T_0^C S_1$ and $T_0^C S_2$ are strongly transversal. Then 0 belongs to the closure of $(S_1 \cap S_2) \setminus \{0\}$.

In view of Theorems 1 and 2, it is natural to ask whether there exists a notion of “tangent cone” that contains both the Boltyanski and Clarke concepts, and is such that the obvious analogue of Theorems 1 and 2 holds. Statement (#) is the simplest necessary condition for such a notion to exist.

*If (#) is true, then this almost surely will lead to a unified version of the finite-dimensional Pontryagin Maximum Principle under very general conditions. I (#{}) is false, then it will almost surely follow that there are at least two noncomparable versions.*