1 Homework assignment No. 5, due on Thursday February 23.

1. In a previous assignment, we defined the maximum of two real numbers $x, y$ to be the real number $x \uparrow y$ given by

$$x \uparrow y = x \text{ if } x \geq y,$$

$$x \uparrow y = y \text{ if } x < y.$$ 

Similarly, we can define the minimum of $x$ and $y$ to be the real number $x \downarrow y$ given by

$$x \downarrow y = y \text{ if } x \geq y,$$

$$x \downarrow y = x \text{ if } x < y.$$ 

Prove or disprove each of the following statements involving the operations $+$, $\times$, $\uparrow$ and $\downarrow$:

i. $(\forall x, y, z \in \mathbb{R}) (x \uparrow (y + z) = (x \uparrow y) + (x \uparrow z) \text{ (distributive law of } \uparrow \text{ with respect to } +))$,

ii. $(\forall x, y, z \in \mathbb{R}) (x \uparrow (y + z) = (x \uparrow y) + (x \uparrow z) \text{ (distributive law of } \uparrow \text{ with respect to } +))$,

iii. $(\forall x, y, z \in \mathbb{R}) (x \uparrow (y \downarrow z) = (x \uparrow y) \downarrow (x \uparrow z) \text{ (distributive law of } \uparrow \text{ with respect to } \downarrow))$,

iv. $(\forall x, z \in \mathbb{R}) (x \uparrow (y \times z) = (x \uparrow y) \times (x \uparrow z) \text{ (distributive law of } \uparrow \text{ with respect to } \times))$,

v. $(\forall x, y \in \mathbb{R}) (x \uparrow y = -((-x) \downarrow (-y)))$.

2. Describe with words, and sketch the picture in the plane, of the region $\mathcal{R}$ given by the equations $|x| \uparrow 1 = 1$, $|y| \uparrow 1 = 1$. (In other words, $\mathcal{R}$ is the set of all points $(x, y)$ such that $|x| \uparrow 1 = 1$ and $|y| \uparrow 1 = 1$.)

3. The division theorem for the integers says that if $a, b$ are integers such that $b \neq 0$ then there exist unique integers $q, r$ such that $a = bq + r$ and $0 \leq r < |b|$. REMARK: In formal language, this says that

$$(\forall a, b \in \mathbb{Z})(b \neq 0 \implies (\exists! q, r \in \mathbb{Z})(a = bq + r \land 0 \leq r < |b|)).$$
i. Suppose a student makes a “small mistake” and states the division theorem as follows:

\[(\forall a, b \in \mathbb{Z}) b \neq 0 \implies (\exists! q, r \in \mathbb{Z})(a = bq + r \land 0 \leq r < |b|)\].

Translate this statement into plain English and decide whether it is true, false, or meaningless.

ii. Suppose a student makes a “small mistake” and states the division theorem as follows:

\[(\forall a, b \in \mathbb{Z})(b \neq 0 \implies (\exists! q, r \in \mathbb{Z})a = bq + r)\].

Translate this statement into plain English and decide whether it is true, false, or meaningless.

3. An integer \(n\) is even iff there exists an integer \(k\) such that \(n = 2k\), and odd iff there exists an integer \(k\) such that \(n = 2k + 1\). (In other words, \(n\) is even if \((\exists k \in \mathbb{Z})n = 2k\), and \(n\) is odd if \((\exists k \in \mathbb{Z})n = 2k + 1\).) Prove that every integer is either even or odd and not both. That is,

\[(\forall n \in \mathbb{Z})((n \text{ is even } \lor n \text{ is odd}) \land \neg (n \text{ is even } \land n \text{ is odd}))\].

(Hint: use the division theorem with \(b = 2\). The two assertions of the division theorem, namely, existence and uniqueness, should correspond to the two parts of our desired conclusion, namely, “\(n\) is even or odd” and “\(n\) is not both even and odd.”)

4. An integer \(n\) is \(0\mod{3}\) iff it is divisible by 3, i.e. if there exists an integer \(k\) such that \(n = 3k\). We say that \(n\) is \(1\mod{3}\) iff there exists an integer \(k\) such that \(n = 3k + 1\). We say that \(n\) is \(2\mod{3}\) iff there exists an integer \(k\) such that \(n = 3k + 2\).

i. Prove that every integer \(n\) satisfies one and only one of the following three possibilities:

- a. \(n\) is \(0\mod{3}\),
- b. \(n\) is \(1\mod{3}\),
- c. \(n\) is \(2\mod{3}\).

HINT: Use the division theorem with \(b = 3\).

ii. Prove that

- a. If \(n\) is \(0\mod{3}\) then \(n^2\) is \(0\mod{3}\).
- b. If \(n\) is \(1\mod{3}\) then \(n^2\) is \(1\mod{3}\).
- c. If \(n\) is \(2\mod{3}\) then \(n^2\) is \(1\mod{3}\).

iii. Prove, using the above results, that if an integer \(n\) is such that \(n^2\) is divisible by 3, then \(n\) is divisible by 3.