1 Functions

Given two sets $A$, $B$, a function from $A$ to $B$ is a rule that assigns to every member $x$ of $A$ a member $f(x)$ of $B$. We write

$$f : A \mapsto B$$

to indicate that $f$ is a function from $A$ to $B$. If $f : A \mapsto B$, then the set $A$ is the domain of the function $f$.

Example 1. Let $A$ be the set of all people, and let $B$ be the set of all women. Then we can define a function $f : A \mapsto B$ (the “mother function”) by stipulating that

$$f(x) = x’s \text{ mother for } x \in A.$$  

Example 2. Let $A$ be the set of all real numbers. Then we can define a function $f : A \mapsto A$ (the “squaring” function) by stipulating that

$$f(x) = x^2 \text{ for } x \in A.$$  

2 Convex functions

Let $I$ be an interval of the real line. A function $f : I \mapsto \mathbb{R}$ is convex if the following is true: whenever $x_1, x_2$ are in $I$, $t \in \mathbb{R}$, and $0 \leq t \leq 1$, the inequality

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$$

holds. (That is, $(\forall x_1 \in I)(\forall x_2 \in I)(\forall t \in \mathbb{R})(0 \leq t \leq 1 \implies f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)).$)
**Example 3.** Let \( f : \mathbb{R} \mapsto \mathbb{R} \) be the squaring function, defined by
\[
f(x) = x^2 \quad \text{for} \quad x \in \mathbb{R}.
\]
Then \( f \) is convex.

**Proof.** We want to prove that
\[
(\forall x_1, x_2, t \in \mathbb{R})(0 \le t \le 1 \implies f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2)),
\]
that is,
\[
(\forall x_1, x_2, t \in \mathbb{R})(0 \le t \le 1 \implies ((1-t)x_1 + tx_2)^2 \le (1-t)x_1^2 + tx_2^2).
\]
Let \( x_1, x_2, t \) be arbitrary real numbers. We want to prove that
\[
0 \le t \le 1 \implies ((1-t)x_1 + tx_2)^2 \le (1-t)x_1^2 + tx_2^2.
\]
Assume that \( 0 \le t \le 1 \). We want to prove that
\[
((1-t)x_1 + tx_2)^2 \le (1-t)x_1^2 + tx_2^2.
\]
We have
\[
((1-t)x_1 + tx_2)^2 = (1-t)x_1^2 + t^2x_2^2 + 2t(1-t)x_1x_2.
\]
On the other hand, the arithmetic-geometric inequality for two numbers tells us that
\[
2x_1x_2 \le x_1^2 + x_2^2.
\]
So
\[
((1-t)x_1 + tx_2)^2 = (1-t)x_1^2 + t^2x_2^2 + 2t(1-t)x_1x_2 \\
\le (1-t)x_1^2 + t^2x_2^2 + (1-t)t(x_1^2 + x_2^2) \\
= ((1-t)^2 + (1-t)t)x_1^2 + t^2 + (1-t)t)x_2^2 \\
= (1-t)((1-t) + t)x_1^2 + t(t + (1-t))x_2^2 \\
= (1-t)x_1^2 + tx_2^2,
\]
that is,
\[
((1-t)x_1 + tx_2)^2 \le (1-t)x_1^2 + tx_2^2,
\]
which is precisely the inequality we wanted to prove.
Example 4. Let \( f : \mathbb{R} \to \mathbb{R} \) be the absolute value function, defined by
\[
f(x) = |x| \quad \text{for} \quad x \in \mathbb{R}.
\]
Then \( f \) is convex.

Proof. We want to prove that
\[
(\forall x_1, x_2, t \in \mathbb{R}) \left( 0 \leq t \leq 1 \implies f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \right),
\]
that is, that
\[
(\forall x_1, x_2, t \in \mathbb{R}) \left( 0 \leq t \leq 1 \implies |(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2| \right).
\]
Let \( x_1, x_2, t \) be arbitrary real numbers. We want to prove that
\[
0 \leq t \leq 1 \implies |(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|.
\]
Assume that \( 0 \leq t \leq 1 \). We want to prove that
\[
|(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|.
\]
It follows from the triangle inequality that
\[
|(1-t)x_1 + tx_2| \leq |(1-t)x_1| + |tx_2|.
\]
On the other hand,
\[
|(1-t)x_1| = |1-t| \cdot |x_1| = (1-t) \cdot |x_1|,
\]
using the identity \( |ab| = |a| \cdot |b| \) and the fact that \( 1-t \geq 0 \) (because \( t \leq 1 \)), and
\[
|tx_2| = |t| \cdot |x_2| = t|x_2|,
\]
using the identity \( |ab| = |a| \cdot |b| \) and the fact that \( t \geq 0 \). Therefore
\[
|(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|,
\]
which is precisely the inequality we wanted to prove.
3 Homework assignment No. 5, due on Thursday February 25

Problem 1. Prove that if \(x\) is a real number such that \(x > 20\) or \(x < 2\) then \(|x - 12| > 7\).

Problem 2. Let \(\mathbb{R}_+\) be the set of all positive real numbers (that is, \(\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}\)). Let \(f : \mathbb{R}_+ \mapsto \mathbb{R}\) be the function given by

\[
f(x) = \frac{1}{x} \quad \text{for } x \in \mathbb{R}_+.
\]

Prove that \(f\) is convex.

Problem 3. Prove that sum of two convex functions is convex. (Precisely: if \(I\) is an interval, and \(f : I \mapsto \mathbb{R}, g : I \mapsto \mathbb{R}\) are convex functions, then \(f + g\) is a convex function.)

Problem 4. A linear function is a function \(f : \mathbb{R} \mapsto \mathbb{R}\) such that, for some real numbers \(a, b\),

\[
f(x) = ax + b \quad \text{for every } x \in \mathbb{R}.
\]

Prove that every linear function is convex.

Problem 5. Prove or disprove: the product of two convex functions is convex.

Problem 6. In this problem, \(P(x)\) and \(Q(x)\) are unknown sentences involving the variable \(x\) and \(U\) is a set. Prove, using the logical rules, that

\[
(\forall x \in U)(P(x) \land Q(x)) \iff (\forall x \in U)P(x) \land (\forall x \in U)Q(x).
\]

Problem 7. In this problem, \(P(x)\) and \(Q(x)\) are unknown sentences involving the variable \(x\) and \(U\) is a set. Show that it cannot be proved, using the logical rules, that

\[
(\forall x \in U)(P(x) \lor Q(x)) \iff (\forall x \in U)P(x) \lor (\forall x \in U)Q(x).
\]