1 Homework assignment no. 3, due on Thursday February 9

1. Prove from the axioms, plus the definitions of 2, 3, and 4, that $2 \times 2 = 4$.

   (NOTE: the definitions of 2, 3, and 4, say that $2 = 1 + 1$, $3 = 2 + 1$, and $4 = 3 + 1$. In the axioms, multiplication is written using $\times$, $\cdot$, or, simply, nothing, so $x \times y$ means the same as $x \cdot y$ and $xy$. Naturally, when $x$ and $y$ are natural numbers written in terms of their decimal expansion, we cannot write their product by just juxtaposing the expressions for $x$ and $y$. For example, we cannot write 22 for “2 times 2” because “22” means “twenty-two”, so we write $2 \times 2$ instead.)

2. Prove that the third equality axiom (Axiom Eq3, transitivity of equality) follows from the other equality axioms (i.e., from Axioms Eq1, Eq2, Eq4). That is, prove Axiom Eq3 using Axioms Eq1, Eq2, Eq4 in your proof.

4. Prove the following statement from the axioms:

   $$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(xy = 0 \Rightarrow (x = 0 \lor y = 0)).$$

   (NOTE: To prove a disjunction $P \lor Q$ you may use the following rule, to be discussed in class next week: if, assuming $\sim P$, you prove $Q$, then you may go to $P \lor Q$. For example, to prove that “today is Tuesday or Thursday” you may proceed as follows: assume today is not Tuesday, and prove that today is Thursday.)

3. For each of the following six statements,

   a. Write the statement in formal language, using “$\in \mathbb{Q}$” for “is rational” and $\notin \mathbb{Q}$ for “is irrational.”

   b. Prove or disprove the statement.

   i. The sum of two irrational numbers is irrational.

   ii. The sum of a rational number and an irrational number is rational.

   iii. The sum of a rational number and an irrational number is irrational.

   iv. The product of two irrational numbers is irrational.

   v. The product of a rational number and an irrational number is rational.

   vi. The product of a rational number and an irrational number is irrational.