1. Let $S$ be the statement "If $A \times B$ are any two sets such that $A \times B = B \times A$, then $A = B$", and let $T$ be the statement "If $A \times B$ are any two nonempty sets such that $A \times B = B \times A$, then $A = B$".

   i. Write $S$ and $T$ in formal language.

   ANSWER $S$ is the statement $$(\forall A, B)((A \times B = B \times A) \implies A = B).$$ and $T$ is the statement $$(\forall A, B)((A \neq \emptyset \land B \neq \emptyset \land A \times B = B \times A) \implies A = B).$$

   ii. Prove or disprove $S$.

   ANSWER $S$ is false, so we disprove it. For this purpose, we give a counterexample. Take $A$ to be any nonempty set (for example, $A = \{\emptyset\}$) and take $B = \emptyset$. Then $A \times B = A \times \emptyset = \emptyset$, and $B \times A = \emptyset \times A = \emptyset$. So $A \times B = B \times A$ but $A \neq B$.

   iii. Prove or disprove $T$.

   ANSWER We prove $T$.

   PROOF:

   Let $A$, $B$ be arbitrary sets.

   We want to prove that $$(\forall A, B)((A \neq \emptyset \land B \neq \emptyset \land A \times B = B \times A) \implies A = B).$$

   Assume that $A \neq \emptyset \land B \neq \emptyset \land A \times B = B \times A$.

   We want to prove $A = B$.

   By the Axiom of Set Equality, “$A = B$” is equivalent to “$\forall x (x \in A \iff x \in B)$.”

   So we will prove that $$(\forall x (x \in A \iff x \in B).$$

   Let $x$ be arbitrary.

   We want to prove that $x \in A \iff x \in B$.

   For this purpose, we will prove that $x \in A \implies x \in B$ and $x \in B \implies x \in A$.

   To prove that $x \in A \implies x \in B$, we assume that $x \in A$ and prove that $x \in B$.

   So assume that $x \in A$.

   Since $B$ is nonempty, there exists a $y$ such that $y \in B$, so we may pick one.

   Then $(x, y) \in A \times B$.

   Since $A \times B = B \times A$, it follows that $(x, y) \in B \times A$.

   Therefore $x \in B$.

   Hence $x \in A \implies x \in B$.

   An identical argument shows that $x \in B \implies x \in A$.

   Then $x \in A \iff x \in B$.

   So $$(\forall x (x \in A \iff x \in B).$$

   Therefore $A = B$.

   Hence $$(\forall A, B)((A \neq \emptyset \land B \neq \emptyset \land A \times B = B \times A) \implies A = B).$$

   So $$(\forall A, B)((A \neq \emptyset \land B \neq \emptyset \land A \times B = B \times A) \implies A = B),$$ as desired.
2. Prove that if \( n, m \) are natural numbers, \( f \) is a bijection from the set \( \{ k \in \mathbb{N} : 1 \leq k \leq n \} \) onto a set \( A \), and \( g \) is a bijection from the set \( \{ k \in \mathbb{N} : 1 \leq k \leq m \} \) onto \( A \), then \( m = n \).

**ANSWER.** Here is the proof.

**PROOF:**

We will do the proof by induction on \( n \). For this purpose, it will be convenient to introduce some notations. We let

\[
N_k = \{ j \in \mathbb{N} : 1 \leq j \leq k \} \quad \text{for} \quad k \in \mathbb{N}.
\]

and we use bij(\( h \)) as an abbreviation for “\( h \) is a bijection.”

Let \( (\forall n)P(n) \) be statement:

\[
(\forall m \in \mathbb{N})((\forall f)(\forall g)(\forall A)((f : N_n \to A \land g : N_m \to A \land \text{bij}(f) \land \text{bij}(g)) \implies m = n)).
\]

We will prove that \( (\forall n)P(n) \), i.e.,

\[
(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})((\forall f)(\forall g)(\forall A)((f : N_n \to A \land g : N_m \to A \land \text{bij}(f) \land \text{bij}(g)) \implies m = n)),
\]

which is exactly what we want to prove.

First we prove \( P(1) \), i.e., we show that if \( f \) is a bijection from \( \{1\} \) to a set \( A \), and \( g \) is a bijection from \( N_{m} \) to \( A \) for an \( m \in \mathbb{N} \), then \( m = 1 \). But this is obvious, because \( A \) has to be \( \{f(1)\} \) (since \( f \) is onto), and then \( g(j) \) must equal \( f(1) \) for every \( j \in N_{m} \); then, since \( g \) is one-to-one, there can only exist one \( j \) in \( N_{m} \), so \( m = 1 \).

We now prove that \( (\forall n \in \mathbb{N})P(n) \implies P(n+1) \).

Let \( n \in \mathbb{N} \) be arbitrary.

Assume that \( P(n) \) is true.

We want to prove \( P(n+1) \). That is, we want to prove that

\[
(\forall m \in \mathbb{N})((\forall f)(\forall g)(\forall A)((f : N_{n+1} \to A \land g : N_{m} \to A \land \text{bij}(f) \land \text{bij}(g)) \implies m = n+1)).
\]

For this purpose, we pick arbitrary \( m \in \mathbb{N} \), an arbitrary set \( A \), and functions \( f, g \), such that \( f \) is a bijection from \( N_{n+1} \) onto \( A \), and \( g \) is a bijection from \( N_{m} \) onto \( A \), and try to prove that \( m = n+1 \).

We consider two different cases, namely, when \( f(n+1) = g(m) \) and when \( f(n+1) \neq g(m) \).

First assume that \( f(n+1) = g(m) \).

Let \( \tilde{A} = A \setminus \{f(n+1)\} \), so \( \tilde{A} = A \setminus \{g(m)\} \). (Recall that, if \( X, Y \) are sets, the difference \( X \setminus Y \) is defined by \( X \setminus Y = \{x : x \in X \land x \notin Y\} \). In particular, \( A \setminus \{f(n+1)\} \) is the set obtained from \( A \) by removing \( f(n+1) \). Define functions \( \tilde{f} : N_{n} \to \tilde{A}, \tilde{g} : N_{m-1} \to \tilde{A}, \) by letting \( \tilde{f}(k) = f(k) \) for \( k \in N_{n} \), \( \tilde{g}(k) = g(k) \) for \( k \in N_{m-1} \). Then \( \tilde{f} \) is a bijection from \( N_{n} \) onto \( \tilde{A} \), and \( \tilde{g} \) is a bijection from \( N_{m-1} \) onto \( \tilde{A} \). It then follows from the inductive assumption that \( n = m - 1 \), and then \( n + 1 = m \), as desired.

We now consider the case when \( f(n+1) \neq g(m) \).

Pick a \( k \in N_{m} \) such that \( g(k) = f(n+1) \). Then \( k \neq m \), because \( g(m) \neq f(n+1) \).

Define \( \hat{g} : N_{m} \to A \) by letting

\[
\hat{g}(j) = g(j) \quad \text{if} \quad j \neq k \land j \neq m,
\]

\[
\hat{g}(k) = g(m),
\]

\[
\hat{g}(m) = g(k).
\]

Then \( \hat{g} \) is a bijection from \( N_{m} \) onto \( A \) such that \( f(n+1) = \hat{g}(m) \). Therefore, with \( \hat{g} \) instead of \( g \), we are in the first case, and then \( m = n + 1 \), completing the proof.
3. Prove or disprove each of the following three statements.

i. If $A, B, C$ are sets, and $f : A \rightarrow B$, $g : B \rightarrow C$, are functions, then, if $f$ and $g$ are injective (i.e., one to one) the composite function $g \circ f : A \rightarrow C$ is injective. (Recall that the composite $g \circ f$ is the function from $A$ to $C$ given by $(g \circ f)(a) = g(f(a))$ for $a \in A$.)

**PROOF.** Assume $f$ and $g$ are injective.

We want to prove that $g \circ f$ is injective, i.e., that

$$(\forall x, y \in A)(x \neq y \implies (g \circ f)(x) \neq (g \circ f)(y)).$$

Let $x, y$ be arbitrary members of $A$.

Assume $x \neq y$

Then $f(x) \neq f(y)$, because $f$ is injective.

Therefore $g(f(x)) \neq g(f(y))$, because $g$ is injective.

So $(g \circ f)(x) \neq (g \circ f)(y)$.

So $g \circ f$ is injective, as desired.

ii. If $A, B, C$ are sets, and $f : A \rightarrow B$, $g : B \rightarrow C$, are functions, then, if $g \circ f$ is injective it follows that $f$ is injective.

**PROOF.** Assume $g \circ f$ is injective.

We want to prove that $f$ is injective, i.e., that

$$(\forall x, y \in A)(x \neq y \implies f(x) \neq f(y)).$$

Let $x, y$ be arbitrary members of $A$.

We want to prove that $f(x) \neq f(y)$.

Assume that $f(x) = f(y)$.

Then $g(f(x)) = g(f(y))$.

So $(g \circ f)(x) = (g \circ f)(y)$.

But $(g \circ f)(x) \neq (g \circ f)(y)$, because $g \circ f$ is injective and $x \neq y$,

So $(g \circ f)(x) = (g \circ f)(y)$ $\land$ $(g \circ f)(x) \neq (g \circ f)(y)$, which is a contradiction.

Hence $f(x) \neq f(y)$.

So $f$ is injective.

iii. If $A, B, C$ are sets, and $f : A \rightarrow B$, $g : B \rightarrow C$, are functions, then, if $g \circ f$ is injective it follows that $g$ is injective.

**ANSWER.** This is not true, so we give a counterexample. Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1, 2\}$.

Define $f : A \rightarrow B$, $g : B \rightarrow C$, by letting $f(1) = 1$, $f(2) = 2$, $g(1) = 1$, $g(2) = 2$, $g(3) = 2$. Then $g \circ f$ is injective, but $g$ is not.
4. The difference of two sets $A$, $B$ is the set $A \setminus B$ given by

$$A \setminus B = \{ x : x \in A \land x \notin B \}.$$ 

The symmetric difference of $A$ and $B$ is the set $A \Delta B$ given by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

**Prove** that

$$(\forall A, B) \quad A \Delta B = (A \cup B) \setminus (A \cap B).$$

**PROOF**

Let $A$, $B$ be arbitrary sets.

We want to prove that $A \Delta B = (A \cup B) \setminus (A \cap B)$.

That is, we want to prove that $(\forall x)(x \in A \Delta B \iff x \in (A \cup B) \setminus (A \cap B))$.

Let $x$ be arbitrary.

We want to prove that $x \in A \Delta B \iff x \in (A \cup B) \setminus (A \cap B)$.

For this purpose, we will prove that $x \in A \Delta B \implies x \in (A \cup B) \setminus (A \cap B)$ and $x \in (A \cup B) \setminus (A \cap B) \implies x \in A \Delta B$.

First, we prove that $x \in A \Delta B \implies x \in (A \cup B) \setminus (A \cap B)$.

Assume that $x \in A \Delta B$.

Then $x \in (A \setminus B) \cup (B \setminus A)$.

Therefore $x \in A \setminus B \lor x \in B \setminus A$.

So we do a proof by cases.

Consider first the case when $x \in A \setminus B$.

Then $x \in A$ and $x \notin B$.

Since $x \in A$, it follows that $x \in A \lor x \in B$, so $x \in A \cup B$.

Since $x \notin B$, it follows that $x \notin A \cap B$ (because if $x$ belonged to $\cap B$ it would follow that $x \in B$).

So $x \in A \cup B \land x \notin A \cap B$.

Therefore $x \in (A \cup B) \setminus (A \cap B)$.

The case when $x \in B \setminus A$ is handled similarly, and we also get $x \in (A \cup B) \setminus (A \cap B)$.

So $x \in (A \cup B) \setminus (A \cap B)$.

Hence $x \in A \Delta B \implies x \in (A \cup B) \setminus (A \cap B)$.

Now we prove that $x \in (A \cup B) \setminus (A \cap B) \implies x \in A \Delta B$.

Assume that $x \in (A \cup B) \setminus (A \cap B)$.

Then $x \in A \cup B$ and $x \notin A \cap B$.

Since $x \in A \cup B$, it follows that $x \in A \lor x \in B$.

So we do a proof by cases.

Assume first that $x \in A$.

Since $x \notin A \cap B$, it follows that $x \notin B$ (because if $x \in B$ then $x \in A \cap B$, since $x \in A$).

Hence $x \in A$ and $x \notin B$.

So $x \in A \setminus B$.

Since $A \Delta B = (A \setminus B) \cup (B \setminus A)$, it follows that $x \in A \Delta B$.

The case when $x \in B$ is handled similarly, and we also get $x \in A \Delta B$.

Therefore $x \in A \Delta B$.

Hence $x \in (A \cup B) \setminus (A \cap B) \implies x \in A \Delta B$, completing our proof.