Superrigidity and Classification Problems

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Classification problems

Open Question

Is the classification problem for the fields of transcendence degree 8 strictly harder than that for the fields of transcendence degree 1?

Open Question

Does there exist a satisfactory classification of the fields of transcendence degree 1 in terms of suitably chosen complete invariants?

Following Friedman-Stanley and Hjorth-Kechris, we will use the theory of Borel equivalence relations to analyze the isomorphism relations on various classes of countable structures and develop a framework for measuring the complexity of possible complete invariants.
Borel reductions

Definition

Let $E$, $F$ be Borel equivalence relations on the standard Borel spaces $X$, $Y$.

(a) $E \leq_B F$ iff there exists a Borel map $f : X \rightarrow Y$ such that

$$x E y \iff f(x) F f(y).$$

In this case, $f$ is called a Borel reduction from $E$ to $F$.

(b) $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.

(c) $E <_B F$ iff both $E \leq_B F$ and $E \nmid_B F$.

Definition

More generally, $f : X \rightarrow Y$ is a Borel homomorphism from $E$ to $F$ iff

$$x E y \implies f(x) F f(y).$$
Milestones

$E_{K_\sigma} = \text{universal } K_\sigma$

$E_1 = \text{hypersmooth}$

$E_0 = \text{hyperfinite}$

$id_\mathbb{R} = \text{smooth}$

$E_\infty = \text{universal countable}$

Countable Borel equivalence relations
**Definition**

$E_0$ is the equivalence relation on $2^\mathbb{N}$ defined by

$$x \ E_0 \ y \quad \text{iff} \quad x(n) = y(n) \text{ for all but finitely many } n.$$

**Definition**

Let $\mathbb{F}_2$ be the free group on two generators. Then $E_\infty$ is the equivalence relation on the powerset $\mathcal{P}(\mathbb{F}_2) = 2^{\mathbb{F}_2}$ defined by

$$X \ E_\infty \ Y \quad \text{iff} \quad \text{there exists } g \in \mathbb{F}_2 \text{ such that } gX = Y.$$
Some essentially countable universal equivalence relations

Theorem (Jackson-Kechris-Louveau)

The isomorphism relation on the space of connected locally finite graphs is essentially countable universal.

Theorem (Thomas-Velickovic)

The isomorphism relation on the space of fields of finite transcendence degree is essentially countable universal.

Theorem (Thomas-Velickovic)

The isomorphism relation on the space of finitely generated groups is countable universal.
Let $\mathbb{F}_m$ be the free group on \{x_1, \ldots, x_m\} and let $\mathcal{G}_m$ be the compact space of normal subgroups of $\mathbb{F}_m$. Since each $m$-generator group can be realised as a quotient $\mathbb{F}_m/N$ for some $N \in \mathcal{G}_m$, we can regard $\mathcal{G}_m$ as the space of $m$-generator groups. There are natural embeddings

$$\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{G}_m \hookrightarrow \cdots$$

and we can regard

$$\mathcal{G} = \bigcup_{m \geq 1} \mathcal{G}_m$$

as the space of f.g. groups.
A slight digression

**Theorem (Champetier-Guirardel)**

If $G$ is a finitely generated group, then the following are equivalent:

- $G$ is a limit group in the sense of Sela.
- $G$ is a limit of free groups in some compact space $G_m$.
- $G$ has the same universal theory as a free group.

**Open Question (Grigorchuk)**

What is the Cantor-Bendixson rank of $G_m$?

**Open Question (Ghys)**

Does there exist a nonatomic $\cong$-invariant ergodic probability measure on $G_m$?
The isomorphism relation

**Theorem (Champetier)**

*The isomorphism relation* \( \cong \) *on the space* \( \mathcal{G} \) *of f.g. groups is a countable Borel equivalence relation.*

The natural action of the countable group \( \text{Aut}(F_m) \) on \( F_m \) induces a corresponding homeomorphic action on the compact space \( \mathcal{G}_m \) of normal subgroups of \( F_m \). Furthermore, each \( \pi \in \text{Aut}(F_m) \) extends to a homeomorphism of the space \( \mathcal{G} \) of f.g. groups.

Clearly if \( N, M \in \mathcal{G}_m \) and there exists \( \pi \in \text{Aut}(F_m) \) such that \( \pi(N) = M \), then \( F_m/N \cong F_m/M \). Unfortunately, the converse does not hold.
The isomorphism relation continued

**Theorem (Tietze)**

If \( N, M \in G_m \), then the following are equivalent:

1. \( \mathbb{F}_m/N \cong \mathbb{F}_m/M \).
2. There exists \( \pi \in \text{Aut}(\mathbb{F}_{2m}) \) such that \( \pi(N) = M \).

**Corollary (Champetier)**

The isomorphism relation \( \cong \) on the space \( G \) of f.g. groups is the orbit equivalence relation arising from the homeomorphic action of the countable group of finitary automorphisms of the free group \( \mathbb{F}_\infty \) on \( \{x_1, x_2, \ldots, x_m, \ldots\} \).
Essentially free countable Borel equivalence relations

Question

Can the isomorphism relation on $\mathcal{G}$ be realised as the orbit equivalence relation of a free Borel action of a suitably chosen countable group?

Question (Jackson-Kechris-Louveau)

Equivalently, is every countable Borel equivalence relation essentially free? In other words, if $E$ is a countable Borel equivalence relation, does there necessarily exist a free countable Borel equivalence relation $F$ such that $E \sim_B F$?
Torsion-free abelian groups of finite rank

**Definition**

An additive subgroup $G \leq \mathbb{Q}^n$ has rank $n$ iff $G$ contains $n$ linearly independent elements.

**Definition**

Let $\cong_n$ denote the isomorphism relation on the standard Borel space $R(\mathbb{Q}^n)$ of torsion-free abelian groups of rank $n$.

Note that if $A, B \in R(\mathbb{Q}^n)$, then

$$A \cong B \iff \exists g \in GL_n(\mathbb{Q}) \ g(A) = B.$$  

In other words, $\cong_n$ is the orbit equivalence relation for the action of $GL_n(\mathbb{Q})$ on the space $R(\mathbb{Q}^n)$. 
Some History

- In 1937, Baer gave a satisfactory classification of the rank 1 groups. (In fact, the isomorphism relation is hyperfinite.)
- In 1938, Kurosh and Malcev independently gave an unsatisfactory classification of the higher rank groups.

Problem (Fuchs 1973)

Characterize the torsion-free abelian groups of rank 2 by invariants.

- In 1998, Hjorth proved that the classification problem for the rank 2 groups was strictly harder than that for the rank 1 groups.

Question (Hjorth-Kečkris 1996)

Is the isomorphism relation for the torsion-free abelian groups of rank 2 countable universal?
The much maligned Malcev-Kurosh invariants

- For each prime $p$, choose some $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^2$ of rank 2

$$A_p = \mathbb{Z}_p v_1 \oplus \mathbb{Z}_p v_2 \text{ for some independent } v_1, v_2 \in \mathbb{Q}_p^2$$

$$= \mathbb{Q}_p v_1 \oplus \mathbb{Z}_p v_2 \text{ for some independent } v_1, v_2 \in \mathbb{Q}_p^2$$

$$= \mathbb{Q}_p^2$$

so that there exists $n \geq 1$ such that $n(\mathbb{Z} \oplus \mathbb{Z}) \leq A_p$ for all $p$.

- Then every torsion-free abelian group of rank 2 has the form

$$A = \bigcap_p [A_p \cap \mathbb{Q}^2]$$

- Furthermore, if $A, B \in R(\mathbb{Q}^2)$ and $g \in GL_2(\mathbb{Q})$, then

$$g \cdot A = B \iff g \cdot A_p = B_p \text{ for every prime } p.$$
Definition

An abelian group $A$ is said to be $p$-local iff $A$ is $q$-divisible for all primes $q \neq p$.

If $A \in R(\mathbb{Q}^2)$ is $p$-local, then $A_q = \mathbb{Q}_q^2$ for all $q \neq p$ and

$$A_p = \mathbb{Q}_p v_1 \oplus \mathbb{Z}_p v_2$$

if $A \not\cong \mathbb{Q} \oplus \mathbb{Q}, \mathbb{Z}_p \oplus \mathbb{Z}_p$.

Theorem

The classification problem for the $p$-local torsion-free abelian groups of rank 2 is Borel bireducible with the orbit equivalence relation of the action of $GL_2(\mathbb{Q})$ on the projective line $\mathbb{Q}_p \cup \{\infty\}$ over the $p$-adics.
Some applications of Superrigidity

Theorem (Hjorth-Thomas 2004)

If $p \neq q$ are distinct primes, then the classification problems for the $p$-local and $q$-local torsion-free abelian groups of rank 2 are incomparable with respect to Borel reducibility.

Theorem (Thomas 2000)

The complexity of the classification problems for the torsion-free abelian groups of rank $n$ increases strictly with the rank $n$.

Corollary

For each $n \geq 1$, the isomorphism relation for the torsion-free abelian groups of rank $n$ is not countable universal.
A slightly embarrassing question

**Question**

*Is the isomorphism relation on the space of torsion-free abelian groups of finite rank countable universal?*

**Answer**

*Clearly not! But why?*
Definition

Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ with invariant ergodic probability measure $\mu$. Then $E$ is strongly universal iff $E \upharpoonright Y$ is universal for every Borel subset $Y \subseteq X$ with $\mu(Y) = 1$.

Open Question

Does there exist a strongly universal countable Borel equivalence relation?
Borel cocycles

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space with invariant ergodic probability measure $\mu$. Suppose that the countable group $H$ has a free Borel action on $Y$ and that

$$f : X \to Y$$

is a Borel homomorphism between the corresponding orbit equivalence relations. Then we can define a Borel cocycle

$$\alpha : G \times X \to H$$

by setting

$$\alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$
The cocycle identity

Note that

\[ f(x) \xrightarrow{\alpha(g,x)} f(g \cdot x) \xrightarrow{\alpha(h,g \cdot x)} f(hg \cdot x) \]

and hence we have the identity:

\[ \alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \mu\text{-a.e. } x \]

In particular, \( f \) is a permutation group homomorphism iff

\[ \alpha(g, x) = \alpha(g) \]

is a group homomorphism.
Cocycle equivalence

\[(X, \mu)\]

\[\beta(g, x) = b(g \cdot x) \alpha(g, x) b(x)^{-1} \quad \mu\text{-a.e } x\]
Popa Superrigidity

- Let $G$ be a countably infinite group and consider the shift action on $2^G = \mathcal{P}(G)$.
- Let $\mu$ be the usual product probability measure on $2^G$.
- Then $\mu$ is $G$-invariant and $G$ acts ergodically on $(2^G, \mu)$.

Theorem (Popa)

Let $\Gamma$ be a countably infinite Kazhdan group and let $G = \Gamma \times S$, where $S$ is an arbitrary countable group. If $H$ is any countable group, then every Borel cocycle

$$\alpha : G \times 2^G \rightarrow H$$

is equivalent to a group homomorphism of $G$ into $H$.
The non-universality proof begins

Let $S$ be a suitably chosen countable group and let

$$G = SL_3(\mathbb{Z}) \times S.$$ 

Let $E$ be the orbit equivalence relation of the action of $G$ on $(2^G, \mu)$. Suppose that

$$f : 2^G \rightarrow \bigsqcup_{n \geq 1} R(\mathbb{Q}^n)$$

is a Borel reduction from $E$ to the isomorphism relation. After deleting a nullset, we can suppose that

$$f : 2^G \rightarrow R(\mathbb{Q}^n)$$

for some fixed $n \geq 1$. 
**The quasi-equality relation**

**Definition**

If \( A, B \in R(\mathbb{Q}^n) \), then \( A \) and \( B \) are said to be **quasi-equal**, written \( A \approx_n B \), iff \( A \cap B \) has finite index in both \( A \) and \( B \).

**Theorem (Thomas)**

The quasi-equality relation \( \approx_n \) is hyperfinite.

For each \( A \in R(\mathbb{Q}^n) \), let \( [A] \) be the \( \approx_n \)-class containing \( A \). We shall consider the induced action of \( GL_n(\mathbb{Q}) \) on

\[
X = \{ [A] \mid A \in R(\mathbb{Q}^n) \}
\]

of \( \approx_n \)-classes. (Of course, \( X \) is not a standard Borel space.)
Stabilizers of $\approx_n$-classes

**Definition**

For each $A \in R(\mathbb{Q}^n)$, the ring of quasi-endomorphisms is

$$\text{QE}(A) = \{ \varphi \in \text{Mat}_n(\mathbb{Q}) \mid (\exists m \geq 1) \ m\varphi \in \text{End}(A) \}.$$ 

Clearly $\text{QE}(A)$ is a $\mathbb{Q}$-subalgebra of $\text{Mat}_n(\mathbb{Q})$; and so there are only countably many possibilities for $\text{QE}(A)$.

**Definition**

$\text{QAut}(A)$ is the group of units of the $\mathbb{Q}$-algebra $\text{QE}(A)$.

**Lemma (Thomas)**

If $A \in R(\mathbb{Q}^n)$, then $\text{QAut}(A)$ is the setwise stabilizer of $[A]$ in $\text{GL}_n(\mathbb{Q})$. 

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Defining the cocycle

For each \( x \in 2^G \), let \( A_x = f(x) \in R(\mathbb{Q}^n) \).

After deleting a nullset and slightly adjusting \( f \), we can suppose that the setwise stabilizer of each \([A_x]\) is a fixed subgroup \( L \leq GL_n(\mathbb{Q}) \).

Note that the quotient group \( H = N_{GL_n(\mathbb{Q})}(L)/L \) acts freely on the corresponding set \( Y = \{[A] \mid QAut(A) = L\} \) of \( \approx_n \)-classes.

Hence we can define a corresponding Borel cocycle

\[
\alpha : G \times 2^G \to H
\]

by setting

\[
\alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot [A_x] = [A_{g \cdot x}].
\]
A suitably chosen $S$

- Let $S$ be a countable simple nonamenable group which does not embed into any of the countably many possibilities for $H$.
- By Popa, after deleting a nullset and slightly adjusting $f$, we can suppose that
  \[ \alpha : G = SL_3(\mathbb{Z}) \times S \to H \]
  is a homomorphism.
- Since $S \leq \ker \alpha$, it follows that $f : 2^G \to R(\mathbb{Q}^n)$ is a Borel homomorphism from the $S$-action on $2^G$ to the hyperfinite quasi-equality $\approx_n$-relation.
- Since $S$ is nonamenable, it follows that $\mu$-almost all $x$ are mapped to a single $\approx_n$-class, which is a contradiction.
Another application of Popa Superrigidity

**Definition**

If $G$ is a countably infinite group, let

$$\mathcal{C}_G = \{ x \in \mathcal{P}(G) | g \cdot x \neq x \text{ for all } 1 \neq g \in G \}$$

and let $F_G$ be the orbit equivalence relation of $G$ on $\mathcal{C}_G$.

It is easily checked that $\mathcal{C}_G$ has $\mu$-measure 1.
Essentially free countable Borel equivalence relations

**Theorem**

If $E$ is a free countable Borel equivalence relation, then there exists a countable group $G$ such that $F_G \not\preceq_B E$.

**Corollary**

$E_\infty$ is not essentially free.

- Suppose that $E$ can be realised by a free Borel $H$-action on $X$ and let $G = \text{SL}_3(\mathbb{Z}) \times S$, where $S$ is any countable simple group which does not embed into $H$.
- Arguing as above, if $f : (2)^G \to X$ is a Borel reduction, then we can suppose that $f$ is $S$-invariant.
- Since $S$ acts ergodically on $(2)^G$, it follows that $\mu$-almost all $x$ are mapped to a single point, which is a contradiction.
Some final open questions

Open Question

Is the isomorphism relation \( \cong_n \) for the torsion-free abelian groups of rank \( n \) essentially free?

Open Question

Suppose that \( E \) is a countable Borel equivalence relation on the standard Borel space \( X \) with invariant ergodic probability measure \( \mu \). Does there always exist a Borel subset \( Y \subseteq X \) with \( \mu(Y) = 1 \) such that \( E \upharpoonright Y \) is essentially free?

If so, then there does not exist a strongly universal countable Borel equivalence relation.