UNIVERSAL BOREL ACTIONS OF COUNTABLE GROUPS

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ABSTRACT. If the countable group G has a nonabelian free subgroup, then there exists a standard Borel G-space such that the corresponding orbit equivalence relation is countable universal. In this paper, we will consider the question of whether the converse also holds.

1. INTRODUCTION

A Borel equivalence relation E on a standard Borel space X is said to be *countable* if every E-class is countable. For example, if G is a countable group and X is a standard Borel G-space, then the corresponding G-orbit equivalence relation E_G^X is a countable Borel equivalence relation. Conversely, by a remarkable result of Feldman-Moore [5], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

Definition 1.1. A countable group G is said to be *action universal* if there exists a standard Borel G-space X such that E_G^X is universal.

Recall that a countable Borel equivalence relation E is said to be *universal* if $F \leq_B E$ for every countable Borel equivalence relation F. (In this case, we will often say that E is *countable universal*.) For example, by Dougherty-Jackson-Kechris [4], if the countable group G has a nonabelian free subgroup, then G is action universal. More precisely, for each countable group G and standard Borel space X, let $E(G, X)$ be the orbit equivalence relation arising from the shift action of G on the standard Borel space X^G. Note that this notation includes the cases

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when X is a finite or a countably infinite standard Borel space. For example, $E(G, 2)$ is the orbit equivalence relation arising from the shift action of G on 2^G.

Theorem 1.2 (Dougherty-Jackson-Kechris [4])**.** *If the countable group G has a nonabelian free subgroup, then $E(G, 2)$ is universal and hence G is action universal.*

No other examples of action universal groups are currently known. On the other hand, the following result is an immediate consequence of the results in Jackson-Kechris-Louveau [8, Section 2].

Theorem 1.3 (Jackson-Kechris-Louveau [8])**.** *If G is a countable amenable group, then G is not action universal.*

This raises the possibility of yet another "dynamic" version of the so-called von Neumann Conjecture that a countable group G is non-amenable if and only if G contains a copy of the free group $\mathbb{F}_2$ on two generators. (The original von Neumann conjecture, which is actually due to Day, was disproved by Ol'shanskii [14] in 1980. For other possible "dynamic" versions, see Jackson-Kechris-Louveau [8, 6.1(D)] and Kechris-Miller [10, Problem 28.14]. We should also mention the remarkable recent result of Gaboriau-Lyons [6] which states that if G is a countable non-amenable group, then there is a free standard Borel G-space X with a G-invariant ergodic probability measure μ such that there exists a free ergodic Borel action of $\mathbb{F}_2$ on (X, μ) with $E_{\mathbb{F}_2}^X \subseteq E_G^X$.)

Question 1.4. Is it true that if G is a countable group, then the following statements are equivalent:

  (i) G is action universal.
  (ii) G contains a nonabelian free subgroup.

A positive answer to Question 1.4 seems extremely unlikely; and in Section 5, we will present some evidence which suggests that free Burnside groups of sufficiently large odd exponent are counterexamples.

Conjecture 1.5. If n is a sufficiently large odd integer, then the free 2-generator Burnside group $B(2, n)$ of exponent n is action universal.

Of course, it is also natural to ask whether *every* countable non-amenable group is action universal. However, in Section 4, we will prove that this is not true. (As we will explain in Section 4, Theorem 1.6 is a consequence of Theorem 4.7.)

Theorem 1.6. *There exists a countable non-amenable group which is not action universal.*

In the final section of this paper, we will switch our attention from universal actions to G-universal actions. Here if G is a countable group and X is a standard Borel G-space, then E_G^X is said to be *G-universal* if $E_G^Z \leq_B E_G^X$ for every standard Borel G-space Z. In [4], Dougherty-Jackson-Kechris proved that if G is any countable group, then $E(G, 2^{\mathbb{N}})$ is G-universal; and in [8], Jackson-Kechris-Louveau proved that if G is any countable group, then $E(G, \mathbb{N})$ is G-universal. However, there are currently no countable groups G for which it is known that $E(G, 2)$ is not G-universal.

By Theorem 1.2, if G has a nonabelian free subgroup, then $E(G, 2)$ is universal and hence $E(G, 2)$ is G-universal. On the other hand, suppose that G is amenable and that X is a standard Borel G-space. By Connes-Feldman-Weiss [3], if μ is *any* Borel probability measure on X, then there exists a G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that the restriction $E_G^X \upharpoonright X_0 = E_G^X \cap (X_0 \times X_0)$ is hyperfinite and it follows that $E_G^X \upharpoonright X_0 \leq_B E(G, 2)$. While these considerations do not rule out the existence of an amenable group G such that $E(G, 2)$ is not G-universal, they suggest that it would be more effective to focus our attention on non-amenable groups with no nonabelian free subgroups. In Section 6, we will prove the following result.

Theorem 1.7. *If G is a simple quasi-finite sofic Kazhdan group, then*

$$E(G, 2) <_B E(G, 3) <_B \cdots <_B E(G, n) <_B \cdots <_B E(G, \mathbb{N}).$$

It is currently not known whether there are any groups satisfying the hypotheses of Theorem 1.7. However, if every hyperbolic group is residually finite, then such groups exist. (For more on the question of the residual finiteness of hyperbolic groups, see Kapovich-Wise [9].)

This paper is organized as follows. In Section 2, we will recall some basic notions from the theory of countable Borel equivalence relations; and we will state an easily

applicable consequence of Popa's Cocycle Superrigidity Theorem which does not
explicitly mention Borel cocycles. In Section 3, we will introduce the notion of
a weakly action universal group; and we will prove that if G is weakly action
universal, then the conjugacy relation $\approx_G$ of G on the space of its subgroups is
not essentially free. In Section 4, we will consider the question of which countable
Borel equivalence relations can be realized up to Borel bireducibility as $\approx_G$ for
some countable group G; and we will prove that there exists an uncountable family
$\{\, G_\alpha \mid \alpha < 2^{\aleph_0} \,\}$ of groups such that the conjugacy relations $\approx_{G_\alpha}$ are pairwise
incomparable with respect to Borel reducuibility. In Section 5, we will prove that if
n is a sufficiently large odd integer and $G = B(2, n)$ is the free 2-generator Burnside
group of exponent n, then $E(G, 2)$ is not essentially free. Finally, in Section 6, we
will switch our attention from universal actions to G-universal actions; and we will
use the remarkable recent work of Bowen [2] on the ergodic theory of sofic groups
to prove Theorem 1.7.

2. Preliminaries

In this section, we will recall some basic notions from the theory of countable
Borel equivalence relations; and we will state an easily applicable consequence of
Popa's Cocycle Superrigidity Theorem which does not explicitly mention Borel
cocycles.

2.1. **Countable Borel equivalence relations.** A detailed development of the
general theory of countable Borel equivalence relations can be found in Jackson-
Kechris-Louveau [8]. Here we will only remind the reader of a few basic notions.

Suppose that E, F be countable Borel equivalence relations on the standard
Borel spaces X, Y respectively. Then a Borel map $\varphi : X \to Y$ is said to be a
homomorphism from E to F if for all x, $y \in X$,

$$x \, E \, y \quad \implies \quad \varphi(x) \, F \, \varphi(y).$$

If φ satisfies the stronger property that for all x, $y \in X$,

$$x \, E \, y \quad \iff \quad \varphi(x) \, F \, \varphi(y),$$

then φ is said to be a *Borel reduction* and we write $E \leq_B F$. If both $E \leq_B F$ and
$F \leq_B E$, then we write $E \sim_B F$ and say that E, F are Borel bireducible. We write

$E <_B F$ if both $E \leq_B F$ and $E \not\leq_B F$. Finally, if there exists a countable-to-one Borel homomorphism $\varphi : X \to Y$ from E to F, then we say that E is *weakly Borel reducible* to F and write $E \leq_B^w F$. In this case, φ is said to be a *weak Borel reduction* from E to F. As expected, a countable Borel equivalence relation E is *weakly universal* if $F \leq_B^w E$ for every countable Borel equivalence relation F. (It is currently not known whether there exists a weakly universal countable Borel equivalence relation which is not universal. For a discussion of this interesting open problem, see Thomas [27].)

Suppose that G is a countable group and that X is a standard Borel G-space; i.e. that there exists a Borel action $(g, x) \mapsto g \cdot x$ of G on X. Then G is said to act *freely* on X if $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$. In this case, we say that X is a *free standard Borel G-space*. If E is a countable Borel equivalence on the standard Borel space X, then E is said to be *free* if there exists a countable group G with a free Borel action on X such that $E_G^X = E$. The countable Borel equivalence relation E is said to be *essentially free* if there exists a free countable Borel equivalence relation F such that $E \sim_B F$.

2.2. **Popa Superrigidity.** The proofs of most of the results in this paper make essential use of Popa's Cocycle Superrigidity Theorem [22]. In this subsection, in order to make the paper intelligible to readers who are unfamiliar with the notions and techniques of superrigidity theory, we will state an easily applicable consequence of Popa's Theorem which does not explicitly mention Borel cocycles. First we need to give two preliminary definitions.

Definition 2.1. Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y and that μ is a Borel probability measure on X. Then the Borel homomorphism $\varphi : X \to Y$ from E to F is said to be μ-*trivial* if there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that φ maps Z into a single F-class. Otherwise, φ is said to be μ-*nontrivial*.

Definition 2.2. If G, H are groups, then the group homomorphism $\pi : G \to H$ is a *virtual embedding* if the kernel $\ker \pi$ is finite.

Throughout this paper, μ_m will denote the usual product probability measure on m^G. The following result is an easy consequence of Popa's Cocycle Superrigidity

Theorem [22]. (For example, see Thomas [26, Section 5].) Here we will only mention that the hypothesis that H is a *free* standard Borel H-space is necessary in order to be able to define a cocycle to which Popa's Theorem can be applied.

Theorem 2.3. *Let Γ be a countably infinite Kazhdan group and let G be a countable group such that $\Gamma \trianglelefteq G$. Suppose that H is any countable group and that Z is a free standard Borel H-space. If $\varphi : m^G \to Z$ is a μ_m-nontrivial Borel homomorphism from $E(G, m)$ to E_H^Z, then there exists*

(i) *a virtual embedding $\pi : G \to H$,*

(ii) *a G-invariant Borel subset $Y \subseteq m^G$ with $\mu_m(Y) = 1$, and*

(iii) *a Borel map $b : m^G \to H$*

such that the "adjusted homomorphism" $\varphi'(y) = b(y)\varphi(y)$ satisfies

$$\varphi'(g \cdot y) = \pi(g) \cdot \varphi'(y)$$

for all $g \in G$ and $y \in Y$.

3. Weakly action universal groups

In this section, we will introduce the notion of a weakly action universal group; and we will prove that if G is weakly action universal, then the conjugacy relation of G on the space of its subgroups is necessarily complicated. (As we will see, the appropriate level of generality for our study turns out to be the class of weakly action universal groups rather than the more obvious class of action universal groups.) Recall that a countable Borel equivalence relation E is weakly universal if for every countable Borel equivalence relation F, there exists a weak Borel reduction (i.e. a countable-to-one Borel homomorphism) from F to E.

Definition 3.1. A countable group G is said to be *weakly action universal* if there exists a standard Borel G-space X such that E_G^X is weakly universal.

The following basic result will play a key role in the remainder of this paper. (It should be pointed out that Theorem 3.2 is an easy consequence of Popa's Cocycle Superrigidity Theorem [22].)

Theorem 3.2 (Thomas [26]). *Suppose that G is a countable group and that X is a standard Borel G-space. If E_G^X is weakly universal, then E_G^X is not essentially free.*

Thus if E_G^X is weakly universal, then there necessarily exist many $x \in X$ such that the point stabilizer $G_x = \{\, g \in G \mid g \cdot x = x \,\}$ is nontrivial. Furthermore, recall that if $g \cdot x = y$, then $G_y = g\, G_x\, g^{-1}$. This suggests that we should study the complexity of the conjugacy relation of G on the space of its subgroups.

Definition 3.3. If G is a countable group, then $\mathrm{Sg}(G)$ denotes the standard Borel space of the subgroups of G and $\approx_G$ denotes the *conjugacy relation* on $\mathrm{Sg}(G)$, which is defined by

$$K \approx_G L \quad \Longleftrightarrow \quad (\exists g \in G)\, L = g\, K\, g^{-1}.$$

If Martin's Conjecture on Turing degree invariant Borel maps is true, then we have the following characterization of the class of weakly action universal groups. However, it should be pointed out that Martin's Conjecture has been an open problem for over 30 years.

Theorem 3.4. *Assuming Martin's Conjecture, if G is a countable group, then the following are equivalent:*

(i) *G is weakly action universal.*

(ii) *$\approx_G$ is weakly universal.*

Before we can state Martin's Conjecture, we must first recall some basic notions from recursion theory. We will follow the usual convention of identifying the powerset $\mathcal{P}(\mathbb{N})$ of the natural numbers with the Cantor space $2^{\mathbb{N}}$, by identifying each subset $A \in \mathcal{P}(\mathbb{N})$ with its characteristic function $\chi_A \in 2^{\mathbb{N}}$. If $A,\, B \in 2^{\mathbb{N}}$, then B is *Turing reducible* to A, written $B \leq_T A$, if there exists an oracle Turing machine which computes χ_B when its oracle tape contains χ_A. Here an oracle Turing machine is a Turing machine with a second "read only" tape, called the oracle tape, upon which we can write the characteristic function of any set $A \in 2^{\mathbb{N}}$, which is called the oracle. (For more details, see Rogers [23].) The Turing equivalence relation $\equiv_T$ on $2^{\mathbb{N}}$ is defined by

$$A \equiv_T B \quad \Longleftrightarrow \quad A \leq_T B \text{ and } B \leq_T A.$$

Finally for each $A \in 2^{\mathbb{N}}$, the corresponding *cone* is $\mathcal{C} = \{\, B \in 2^{\mathbb{N}} \mid A \leq_T B \,\}$. (When studying the Turing equivalence relation, the set of cones plays an analogous role to that played by the full-measure subsets in ergodic theory.)

By Martin's Conjecture, we mean the following special case of a more general conjecture (also known as the 5th Victoria Delfino Problem) which was formulated by Martin in Kechris-Moschovakis [11].

Martin's Conjecture. *If $\varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

  (a) *There exists a cone $\mathcal{C} \subseteq 2^{\mathbb{N}}$ such that φ maps $\mathcal{C}$ into a single $\equiv_T$-class.*
  (b) *There exists a cone $\mathcal{C} \subseteq 2^{\mathbb{N}}$ such that $x \leq_T \varphi(x)$ for all $x \in \mathcal{C}$.*

Proof of Theorem 3.4. It is clear that (ii) implies (i). Conversely, suppose that X is a standard Borel G-space such that E_G^X is weakly universal and suppose that $\approx_G$ is *not* weakly universal. Consider the Borel map $\varphi : X \to \mathrm{Sg}(G)$ defined by

$$\varphi(x) = G_x = \{g \in G \mid g \cdot x = x\}.$$

Then φ is a Borel homomorphism from E_G^X to $\approx_G$. Let $\psi : 2^{\mathbb{N}} \to X$ be a weak Borel reduction from $\equiv_T$ to E_G^X and let $\theta = \varphi \circ \psi$. Assuming Martin's Conjecture, by Thomas [27, Theorem 1.4], since θ is a Borel homomorphism from $\equiv_T$ to $\approx_G$ and $\approx_G$ is not weakly universal, there exists a cone $\mathcal{C} \subseteq 2^{\mathbb{N}}$ such that θ maps $\mathcal{C}$ into a single $\approx_G$-class. (Here it should be emphasized that the currently known "proof" of Thomas [27, Theorem 1.4] makes essential use of Martin's Conjecture.) Hence, after slightly adjusting ψ if necessary, we can suppose that there exists a fixed subgroup $K \leqslant G$ such that $G_{\psi(A)} = K$ for all $A \in \mathcal{C}$. For later use, note that $\equiv_T \restriction \mathcal{C}$ is weakly universal. (For example, see Thomas [27, Observation 2.3].)

Let $X_0 = \{\, x \in X \mid G_x = K \,\}$. If $x, y \in X_0$ and $x \, E_G^X \, y$, then there exists an element $g \in G$ such that $g \cdot x = y$. Since

$$gKg^{-1} = gG_x g^{-1} = G_y = K,$$

it follows that $g \in N_G(K)$. Furthermore, if $g' \in N_G(K)$ also satisfies $g' \cdot x = y$, then $g'K = gK$. Thus $E_G^X \restriction X_0$ can be realized as the orbit equivalence relation of the corresponding free Borel action of $\Delta = N_G(K)/K$ on X_0. Applying Theorem 3.2, it follows that $E_G^X \restriction X_0$ is not weakly universal. But this is a contradiction, since $\equiv_T \restriction \mathcal{C} \leq_B^w E_G^X \restriction X_0$ and $\equiv_T \restriction \mathcal{C}$ is weakly universal. $\square$

Unfortunately, as we mentioned earlier, it is currently not known whether Martin's Conjecture is true. In the remainder of this section, we will prove the following weak version of Theorem 3.4.

Theorem 3.5. *If $\approx_G$ is essentially free, then G is not weakly action universal.*

Question 3.6. Does the converse of Theorem 3.5 hold?

The proof of Theorem 3.5 makes use of Popa Superrigidity, together with the following result.

Theorem 3.7 (Andretta-Camerlo-Hjorth [1]). *If the countable group G has a free nonabelian subgroup, then $\approx_G$ is countable universal.*

Proof of Theorem 3.5. Suppose that G is a weakly action universal group such that $\approx_G$ is essentially free. Then there exists a countable group H and a free standard Borel H-space Z such that $\approx_G \sim_B E_H^Z$. Let $\varphi : \mathrm{Sg}(G) \to Z$ be a Borel reduction from $\approx_G$ to E_H^Z. For later use, let L be a finitely generated group with no nontrivial finite normal subgroups such that L does not embed into H. (To see that such a group L exists, recall that there exist uncountably many finitely generated groups up to isomorphism and that H has only countably many finitely generated subgroups. Hence there exists a finitely generated group L_0 which does not embed into H and we can let L be the free product $\mathbb{Z} * L_0$.) Let $\Gamma = SL_3(\mathbb{Z}) \times L$ and note that Γ also has no nontrivial finite normal subgroups.

Next let X be a standard Borel G-space such that E_G^X is weakly universal and let $\psi : 2^\Gamma \to X$ be a weak Borel reduction from $E(\Gamma, 2)$ to E_G^X. Let $\sigma : X \to \mathrm{Sg}(G)$ be the Borel homomorphism defined by $\sigma(x) = G_x$ and let $\theta : 2^\Gamma \to Z$ be the Borel homomorphism from $E(\Gamma, 2)$ to E_H^Z defined by $\theta = \varphi \circ \sigma \circ \psi$. Applying Theorem 2.3, since Γ does not embed into H, there exists a Borel subset $Y \subseteq 2^\Gamma$ with $\mu_2(Y) = 1$ such that θ maps Y into a single E_H^Z-class; and this implies that $\sigma \circ \psi$ maps Y into a single conjugacy class of subgroups of G. Hence, after slightly adjusting ψ if necessary, we can suppose that there exists a fixed subgroup $K \leqslant G$ such that $G_{\psi(y)} = K$ for all $y \in Y$. Let $X_0 = \{x \in X \mid G_x = K\}$. Then $E_G^X \restriction X_0$ can be realized as the orbit equivalence relation of the corresponding free Borel action of the quotient group $\Delta = N_G(K)/K$. Clearly the weak Borel reduction $\psi \restriction Y$ from

$E(\Gamma, 2) \restriction Y$ to $E_G^X \restriction X_0$ can be extended to a μ_2-nontrivial Borel homomorphism from $E(\Gamma, 2)$ to $E_G^X \restriction X_0$. Hence, by Theorem 2.3, there exists an embedding $\pi : \Gamma \to \Delta$. In particular, since $SL_3(\mathbb{Z}) \leqslant \Gamma$, it follows that $\Delta = N_G(K)/K$ has a nonabelian free subgroup; and this implies that that G also has a nonabelian free subgroup. Applying Theorem 3.7, it follows that $\approx_G$ is countable universal and hence $\approx_G$ is not essentially free, which is a contradiction. $\qquad\square$

4. THE CONJUGACY RELATION ON THE SPACE OF SUBGROUPS

In this section, we will consider the question of which countable Borel equivalence relations can be realized up to Borel bireducibility as $\approx_G$ for some countable group G. As we mentioned earlier, Andretta-Camerlo-Hjorth [1] have shown that if G has a free nonabelian subgroup, then $\approx_G$ is countable universal. On the other hand, by Jackson-Kechris-Louveau [8, Proposition 2.13], if G is amenable, then $\approx_G$ is Fréchet amenable. (For the definition of Fréchet amenability, see Jackson-Kechris-Louveau [8, Section 2.4]. It is currently not known whether every Fréchet amenable countable Borel equivalence relation is hyperfinite.) The main result of this section provides many examples of groups G such that $\approx_G$ is neither Fréchet amenable nor countable universal. However, we should point out that the following fundamental question remains open.

Question 4.1. Suppose that E is any countable Borel equivalence relation. Does there necessarily exist a countable group G such that $\approx_G \sim_B E$?

We will begin with some basic observations concerning the relative complexity of the conjugacy relations $\approx_G$, $\approx_H$ for various pairs G, H of groups.

Lemma 4.2. *If G, H are countable groups and there exists a surjective homomorphism $\pi : G \to H$, then $\approx_H \leqslant_B \approx_G$.*

Proof. Let $\varphi : \mathrm{Sg}(H) \to \mathrm{Sg}(G)$ be the map defined by $\varphi(K) = \pi^{-1}(K)$. Then φ is a Borel reduction from $\approx_H$ to $\approx_G$. $\qquad\square$

Next recall that a subgroup H of a group G is said to be *malnormal* if whenever $g \in G \smallsetminus H$, then $gHg^{-1} \cap H = 1$.

Lemma 4.3. *If H is a malnormal subgroup of G, then $\approx_H \leqslant_B \approx_G$.*

Proof. The inclusion map $\mathrm{Sg}(H) \hookrightarrow \mathrm{Sg}(G)$ is a Borel reduction from $\approx_H$ to $\approx_G$.   □

Question 4.4. Does there exist a pair of countable groups $H \leqslant G$ such that $\approx_H \nleq_B \approx_G$?

Remark 4.5. By Theorem 3.7, if $H \leqslant G$ is such a pair, then G has no nonabelian free subgroups.

Finally if H, C are any groups, then C wr H denotes the *(restricted) wreath product* of H and C, which is defined as follows. For each function $f : H \to C$, the *support* $\sigma(f)$ is defined to be

$$\sigma(f) = \{x \in H \mid f(x) \neq 1\};$$

and the corresponding *base group* is defined to be

$$B = \{f : H \to C \mid \sigma(f) \text{ is finite }\},$$

equipped with pointwise multiplication; i.e., if f, $g \in B$, then

$$(fg)(x) = f(x)g(x)$$

for all $x \in H$. There is a natural action of H on B defined by

$$(a \cdot f)(x) = f(a^{-1}x)$$

for all $f \in B$ and a, $x \in H$; and C wr H is the corresponding semidirect product $B \rtimes H$. For each $x \in H$, let

$$C_x = \{ f \in B \mid f(y) = 1 \text{ for all } x \neq y \in H \}.$$

Then $B = \bigoplus_{x \in H} C_x$; and $aC_xa^{-1} = C_{ax}$ for all a, $x \in H$.

Lemma 4.6. *If H, C are countable groups and $G = C$ wr H is the corresponding (restricted) wreath product, then $E(H, 2) \leq_B \approx_G$.*

Proof. Let $B = \bigoplus_{x \in H} C_x$ be the base group of $G = C$ wr H; and for each subset $A \subseteq H$, let K_A be the subgroup of B defined by $K_A = \bigoplus_{a \in A} C_a$. Suppose that $g = hb \in G$ is any element, where $h \in H$ and $b \in B$. Since $K_A \trianglelefteq B$, we have that

$$gK_Ag^{-1} = hbK_Ab^{-1}h^{-1} = hK_Ah^{-1} = K_{hA}.$$

It follows that the map $A \mapsto K_A$ is a Borel reduction from $E(H, 2)$ to $\approx_G$.   □

The remainder of this section will be devoted to the proof of the following result.

Theorem 4.7. *There exists an uncountable family $\{ G_\alpha \mid \alpha < 2^{\aleph_0} \}$ of countable groups such that for all $\alpha < \beta < 2^{\aleph_0}$,*

(i) *$\approx_{G_\alpha}$ is essentially free;*

(ii) *$\approx_{G_\alpha}$ is not Fréchet amenable; and*

(ii) *$\approx_{G_\alpha}$ and $\approx_{G_\beta}$ are incomparable with respect to Borel reducibility.*

Remark 4.8. Note that each G_α satisfies the requirements of Theorem 1.6. To see this, first notice that since $\approx_{G_\alpha}$ is not Fréchet amenable, it follows that G is not amenable. (Here we are applying Jackson-Kechris-Louveau [8, Proposition 2.13].) Secondly, by Theorem 3.5, since $\approx_{G_\alpha}$ is essentially free, it follows that G_α is not weakly action universal.

Each G_α will have the form $C_2 \operatorname{wr} \Gamma_\alpha$, where C_2 is the cyclic group of order 2 and Γ_α is a suitably chosen simple quasi-finite group. Here an infinite group Γ is said to be *quasi-finite* if every proper subgroup of Γ is finite. It is easily shown that every abelian quasi-finite group is isomorphic to a quasi-cyclic group C_{p^∞} for some prime p. (See Ol'shanskii [15, Theorem 7.5].) However, it was a long outstanding problem whether there existed a nonabelian quasi-finite group. This problem was finally solved by Ol'shanskii in his celebrated papers [12, 13]. A clear account of this work can be found in Ol'shanskii [15]. The following result will play an essential role in the proofs of both Theorem 4.7 and Theorem 1.7.

Proposition 4.9 (Thomas [26]). *Suppose that Γ is a simple quasi-finite group and that X is a standard Borel Γ-space. If $Z = \{ x \in X \mid \Gamma_x = 1 \}$ is the free part of the action, then $E_\Gamma^X \sim_B E_\Gamma^X \upharpoonright Z$.*

The following result is implicitly contained in Ol'shanskii [17]. (For more details, see Ozawa [19].)

Theorem 4.10 (Ol'shanskii [17]). *If H is a noncyclic torsion-free hyperbolic group, then H has a family $\{ \Gamma_\alpha = H/N_\alpha \mid \alpha < 2^{\aleph_0} \}$ of uncountably many pairwise nonisomorphic simple quasi-finite quotient groups.*

Remark 4.11. Suppose that H is a noncyclic torsion-free hyperbolic Kazhdan group. (For example, we can let H be a co-compact lattice in $Sp(n, 1)$ for some

$n \geq 2$. See de la Harpe-Valette [7].) Then each $\Gamma_\alpha = H/N_\alpha$ is also a Kazhdan group and hence is non-amenable. Applying Proposition 4.9 and Theorem 3.2, it follows that Γ_α is not weakly action universal. Thus each Γ_α also satisfies the requirements of Theorem 1.6. Of course, since Γ_α is quasi-finite, it follows that $\mathrm{Sg}(\Gamma_\alpha)$ is countable and hence $\approx_{\Gamma_\alpha}$ is smooth.

Proof of Theorem 4.7. Let H be a noncyclic torsion-free hyperbolic Kazhdan group and let $\{\,\Gamma_\alpha = H/N_\alpha \mid \alpha < 2^{\aleph_0}\,\}$ be a family of uncountably many pairwise nonisomorphic simple quasi-finite quotient groups. For each $\alpha < 2^{\aleph_0}$, let $G_\alpha = C_2 \operatorname{wr} \Gamma_\alpha$, where C_2 is the cyclic group of order 2.

To see that each $\approx_{G_\alpha}$ is not Fréchet amenable, first notice that each Γ_α is an infinite Kazhdan group and thus is non-amenable. Hence, by Jackson-Kechris-Louveau [8, Proposition 2.14], $E(\Gamma_\alpha, 2)$ is not Fréchet amenable. By Lemma 4.6, $E(\Gamma_\alpha, 2) \leq_B \approx_{G_\alpha}$ and this implies that $\approx_{G_\alpha}$ is not Fréchet amenable.

Next we will show that each $\approx_{G_\alpha}$ is essentially free. Fix some $\alpha < 2^{\aleph_0}$. In order to simplify notation, let $G = G_\alpha$ and $\Gamma = \Gamma_\alpha$. Let $\pi : G \to \Gamma$ be the canonical surjective homomorphism. Then $\mathrm{Sg}(G) = X \sqcup Y \sqcup Z$, where

- $X = \{H \leqslant G \mid \pi[H] = \Gamma\}$;
- $Y = \{H \leqslant G \mid \pi[H] \text{ is a nontrivial finite subgroup of } \Gamma\}$; and
- $Z = \{H \leqslant G \mid \pi[H] = 1\}$.

We will successively analyze the Borel complexity of $\approx_G$ restricted to each of the above Borel subsets of $\mathrm{Sg}(G)$. From now on, let $B = \bigoplus_{x \in \Gamma} C_x$ be the base group of $G = C_2 \operatorname{wr} \Gamma$, so that $G = B \rtimes \Gamma$.

Claim 4.12. $\approx_G \restriction X$ *is smooth.*

Proof of Claim 4.12. Suppose that $H \in X$ and let $g = \gamma b \in G$ be any element, where $\gamma \in \Gamma$ and $b \in B$. Since $\pi[H] = \Gamma$, there exists an element $c \in B$ such that $h = \gamma c \in H$. It follows that

$$g(H \cap B)g^{-1} = \gamma(H \cap B)\gamma^{-1} = h(H \cap B)h^{-1} = H \cap B.$$

Thus $H \cap B \trianglelefteq G$. Also notice that since $H/H \cap B \cong \Gamma$, it follows that H is finitely generated over $H \cap B$. Hence there exist only countably many $H' \in X$ such that

$H' \cap B = H \cap B$. It follows that if $\equiv$ is the equivalence relation defined on X by

$$H \equiv H' \quad \Longleftrightarrow \quad H \cap B = H' \cap B,$$

then $\equiv$ is a smooth countable Borel equivalence relation. Since $\approx_G \restriction X \subseteq \equiv$, it follows that $\approx_G \restriction X$ is also smooth. (For example, see Thomas [28, Lemma 2.1].) $\quad\square$

Claim 4.13. $\approx_G \restriction Y$ *is smooth.*

Proof of Claim 4.13. Let $\mathcal{F}$ be a set of representatives of the countably many conjugacy classes of nontrivial finite subgroups of Γ; and for each $F \in \mathcal{F}$, let

$$Y_F = \{H \in Y \mid \pi[H] = F\}.$$

Then clearly $\approx_G \restriction Y$ is Borel bireducible with $\approx_G \restriction \bigsqcup_{F \in \mathcal{F}} Y_F$. Hence it is enough to prove that each $\approx_G \restriction Y_F$ is smooth. Fix some $F \in \mathcal{F}$ and suppose that $H \in Y_F$. Since $H/H \cap B \cong F$ is finite, it follows that there exist only countably many $H' \in Y_F$ such that $H' \cap B = H \cap B$. Hence if $\sim$ is the equivalence relation on Y_F defined by

$$H \sim H' \quad \Longleftrightarrow \quad (\exists \gamma \in N_\Gamma(F))\ \gamma(H \cap B)\gamma^{-1} = H' \cap B,$$

then $\sim$ is a countable Borel equivalence relation. Since Γ is a simple quasi-finite group, it follows that $N_\Gamma(F)$ is a finite subgroup of Γ and hence $\sim$ is smooth. Thus it is enough to show that $\approx_G \restriction Y_F \subseteq \sim$. To see this, let $H,\, H' \in Y_F$ and suppose that $gHg^{-1} = H'$. Let $g = \gamma b$, where $\gamma \in \Gamma$ and $b \in B$. Then clearly $\gamma \in N_\Gamma(F)$ and since

$$H' \cap B = g(H \cap B)g^{-1} = \gamma(H \cap B)\gamma^{-1},$$

it follows that $H \sim H'$. $\quad\square$

Using Claim 4.12 and Claim 4.13, it follows that $\approx_G$ is Borel bireducible with $\approx_G \restriction Z$ and thus it only remains to analyze the Borel complexity of $\approx_G \restriction Z$. Suppose that $H \in Z$. Let $g = \gamma b \in G$ be any element, where $\gamma \in \Gamma$ and $b \in B$. Since $H \leqslant B$, it follows that $gHg^{-1} = \gamma H \gamma^{-1}$. Thus $\approx_G \restriction Z$ is the orbit equivalence relation induced by the conjugacy action of the simple quasi-finite Γ; and applying Proposition 4.9, it follows that $\approx_G \restriction Z$ is essentially free. This completes the proof that $\approx_{G_\alpha}$ is essentially free.

Finally we will prove that if $\alpha \neq \beta$, then $\approx_{G_\alpha}$ and $\approx_{G_\beta}$ are incomparable with respect to Borel reducibility. Suppose that $\approx_{G_\alpha} \leq_B \approx_{G_\beta}$. By Lemma 4.6, we have that $E(\Gamma_\alpha, 2) \leq_B \approx_{G_\alpha}$ and hence $E(\Gamma_\alpha, 2) \leq_B \approx_{G_\beta}$. Furthermore, combining Proposition 4.9 with the argument in the previous paragraph, it follows that there exists a free standard Borel Γ_β-space Z' such that $E_{\Gamma_\beta}^{Z'} \sim_B \approx_{G_\beta}$. Let $\varphi : 2^{\Gamma_\alpha} \to Z'$ be a Borel reduction from $E(\Gamma_\alpha, 2)$ to $E_{\Gamma_\beta}^{Z'}$. Applying Theorem 2.3, there exists an embedding $\theta : \Gamma_\alpha \to \Gamma_\beta$; and since Γ_β is quasi-finite, it follows that θ is an isomorphism, which is a contradiction. This completes the proof of Theorem 4.7. $\qquad\square$

5. Free Burnside Groups

In this section, we will present some evidence which supports the conjecture that free Burnside groups of sufficiently large odd exponent are action universal. The following result implies that this conjecture is equivalent to the statement that if n is a sufficiently large odd integer, then there exists an action universal group H of exponent n.

Proposition 5.1. *Let n be a sufficiently large odd integer and let $2 \leq m \leq \omega$. If $G = B(m, n)$ is the free m-generator Burnside group of exponent n and H is any countable group of exponent n, then:*

   (i) *$\approx_H \leq_B \approx_G$; and*
   (ii) *if X is any standard Borel space, then $E(H, X) \leq_B E(G, X)$.*

Proof. If $K = B(\omega, n)$ is the free Burnside group of exponent n on countably many generators, then there exists a surjective homomorphism $\varphi : K \to H$. Hence, by Lemma 4.2, it follows that $\approx_H \leq_B \approx_K$. By Ol'shanskii-Sapir [18, Lemma 4.11] and Sonkin [24, Section 4], there exists a malnormal subgroup L of G such that $L \cong K$; and hence, applying Lemma 4.3, it follows that $\approx_K \leq_B \approx_G$. Thus $\approx_H \leq_B \approx_G$. Similarly, applying Dougherty-Jackson-Kechris [4, Section 1], it follows that if X is any standard Borel space, then $E(H, X) \leq_B E(G, X)$. $\qquad\square$

Most of this section will be devoted to the proof of the following result.

Theorem 5.2. *Let n be a sufficiently large odd integer and let $2 \leq m \leq \omega$. If $G = B(m, n)$ is the free m-generator Burnside group of exponent n, then $E(G, 2)$ is not essentially free.*

Before beginning the proof of Theorem 5.2, we first derive the following easy consequence.

Corollary 5.3. *Suppose that n is a sufficiently large odd composite integer and that $2 \leq m \leq \omega$. If $G = B(m, n)$ is the free m-generator Burnside group of exponent n, then $\approx_G$ is not essentially free.*

Proof. By Proposition 5.1, it is enough to prove that if n is a sufficiently large odd composite number and $G = B(3, n)$ is the free 3-generator Burnside group of exponent n, then $\approx_G$ is not essentially free. Let p be a prime factor of n, chosen so that $n_0 = n/p$ is still sufficiently large. Let $W = C_p \mathrm{wr} B(2, n_0)$. Then W is a 3-generator group of exponent n and hence there exists a surjective homomorphism $G \to W$. It follows that $\approx_W \leq_B \approx_G$. Also, by Lemma 4.6, we have that $E(B(2, n_0), 2) \leq \approx_W$. Hence Theorem 5.2 implies that $\approx_G$ is not essentially free. $\square$

Of course, Corollary 5.3 should also be true if n is a sufficiently large prime. The following result will be used in the proof of Theorem 5.2.

Theorem 5.4 (Ol'shanskii [16])**.** *If H is a noncyclic torsion-free hyperbolic group, then there there exists a natural number n_H such that the group H/H^n is infinite for every odd $n \geq n_H$.*

We will also make use of the following result of Ol'shanskii [15, Theorem 28.7]. (The groups given by Theorem 4.10 have infinite exponent and hence cannot be used in the proof of Theorem 5.2.)

Theorem 5.5 (Ol'shanskii [15])**.** *For every sufficiently large odd integer n, there exists a family $\{G_\alpha \mid \alpha < 2^{\aleph_0}\}$ of pairwise nonisomorphic infinite simple groups such that for each $\alpha < 2^{\aleph_0}$,*

   (a) *G_α is a group of exponent n; and*
   (b) *every nontrivial proper subgroup of G_α is cyclic of order dividing n.*

Remark 5.6. While Ol'shanskii does not state explicitly that each G_α is simple, this follows easily from the fact that the centralizer of each nontrivial element of G_α is cyclic. (See Ol'shanskii [15, Theorem 26.5].) For suppose that N is a nontrivial proper normal subgroup of G_α. Then N is clearly finite; and by considering the action of G_α on N via conjugation, it follows that N must be contained in the center of G_α, which is a contradiction.

Proof of Theorem 5.2. Let H be a noncyclic torsion-free hyperbolic Kazhdan group. By Theorem 5.4, we can suppose that the group $K = H/H^n$ is infinite. Thus K is an infinite Kazhdan group of exponent n. Let $\{G_\alpha \mid \alpha < 2^{\aleph_0}\}$ be the family of pairwise nonisomorphic simple groups of exponent n given by Theorem 5.5. Of course, it is clear that each G_α is a 2-generator group.

Let K be a d-generator group. By Ol'shanskii [15, Theorem 39.1], $B(2, n)$ contains a subgroup which is isomorphic to the free Burnside group $B(d+2, n)$ on $d+2$ generators. Hence, by Dougherty-Jackson-Kechris [4, Proposition 1.5], we have that $E(B(d+2, n), 2) \leq_B E(B(2, n), 2)$ and so it is enough to show that $E(B(d+2, n), 2)$ is not essentially free. To see this, first notice that for each $\alpha < 2^{\aleph_0}$, the group $L_\alpha = K \times G_\alpha$ is a homomorphic image of $B(d+2, n)$; and hence, by Dougherty-Jackson-Kechris [4, Proposition 1.4], we have that $E(L_\alpha, 2) \leq_B E(B(d+2, n), 2)$. Now suppose that $E(B(d+2, n), 2) \sim_B E_\Delta^X$, where Δ is a countable group and X is a free standard Borel Δ-space. Then for each $\alpha < 2^{\aleph_0}$, there exists a Borel reduction $\varphi_\alpha : 2^{L_\alpha} \to X$ from $E(L_\alpha, 2)$ to E_Δ^X; and hence, by Theorem 2.3, there exists a virtual embedding $\pi_\alpha : L_\alpha \to \Delta$. Since G_α is an infinite simple group and $\ker \pi_\alpha$ is finite, it follows that $\pi_\alpha \upharpoonright G_\alpha$ is an embedding. Since Δ has only countably many 2-generator subgroups, it follows that there exist $\alpha \neq \beta$ such that $\pi_\alpha[G_\alpha] = \pi_\beta[G_\beta]$, which contradicts the fact that G_α and G_β are nonisomorphic. $\qquad\square$

6. G-UNIVERSAL ACTIONS

In this final section, we will switch our attention from universal actions to G-universal actions. Recall that Jackson-Kechris-Louveau [8, Theorem 5.4] have shown that if G is any countable group, then $E(G, \mathbb{N})$ is G-universal. On the other hand, there are currently no countable groups G for which it is known that $E(G, 2)$

is not G-universal. We will initially focus on the free parts of the various shift actions.

Definition 6.1. For each countable group G and standard Borel space X, the *free part* of X^G is

$$(X)^G = \{\, p \in X^G \mid g \cdot p \neq p \text{ for all } 1 \neq g \in G \,\};$$

and $F(G, X) = E(G, X) \restriction (X)^G$ is the corresponding orbit equivalence relation.

By Jackson-Kechris-Louveau [8, Section 5.1], if G is any countable group and Z is a free standard Borel G-space, then $E_G^Z \leq_B F(G, \mathbb{N})$. On the other hand, by Jackson-Kechris-Louveau [8, Theorem 3.17], letting $\mathbb{F}_2$ denote the free group on two generators, we have that $F(\mathbb{F}_2, 2) \sim_B F(\mathbb{F}_2, \mathbb{N})$. The following is the main result of this section.

Theorem 6.2. *If G is an co-hopfian sofic Kazhdan group with no nontrivial finite normal subgroups, then*

$$F(G, 2) <_B F(G, 3) <_B \cdots <_B F(G, n) <_B \cdots <_B F(G, \mathbb{N}).$$

Here a group G is said to be *co-hopfian* if every embedding $\pi : G \to G$ is an automorphism. A clear account of the basic theory of *sofic* groups can be found in Pestov [22]. It is an important open problem whether every group is sofic. For our purposes, it is enough to mention the following points:

- Every residually finite group is sofic; and, in particular, it follows that finitely generated linear groups are sofic. The results of Ol'shanskii [17] imply that if every hyperbolic group is residually finite, then there exist groups satisfying the hypotheses of Theorem 1.7.
- It is currently not known whether there exists an infinite finitely generated simple sofic group.

Corollary 6.3. *If $G = SL_3(\mathbb{Z})$, then*

$$F(G, 2) <_B F(G, 3) <_B \cdots <_B F(G, n) <_B \cdots <_B F(G, \mathbb{N}).$$

Proof of Corollary 6.3. It is well-known that $SL_3(\mathbb{Z})$ is a Kazhdan group with no nontrivial finite normal subgroups; and by the preceeding remarks, $SL_3(\mathbb{Z})$ is sofic. Finally, by Steinberg [25, Theorem 6], $SL_3(\mathbb{Z})$ is also co-hopfian. $\square$

Next we note that Theorem 1.7 is an easy consequence of Theorem 6.2.

Proof of Theorem 1.7. If G is a simple quasi-finite sofic Kazhdan group, then G clearly satisfies the hypotheses of Theorem 6.2; and by Proposition 4.9, we have that $E(G, X) \sim_B F(G, X)$ for every standard Borel space X. $\square$

The following notion will play a key role in the proof of Theorem 6.2.

Definition 6.4. Let G be a countable group and let Z be a standard Borel G-space. If I is a countable set, then an I-*generator* is a partition $Z = \bigsqcup_{i \in I} A_i$ into Borel subsets such that $\{g \cdot A_i \mid i \in I, g \in G\}$ separates points. (Equivalently, $\{g \cdot A_i \mid i \in I, g \in G\}$ generates the σ-algebra of Borel subsets of Z.)

Example 6.5. Consider the shift action of the countable group G on the standard Borel space $Z = (m)^G$; and for each $i \in m$, let

$$B_i = \{x \in (m)^G \mid x(1) = i\}.$$

Then $\beta = (B_0, B_1, \cdots, B_{m-1})$ is an m-generator

Recall that if G is a countable group and $m \geq 2$, then μ_m denotes the usual product probability measure on m^G. It is easily checked that $\mu_m((m)^G) = 1$.

Lemma 6.6. *If G is a co-hopfian Kazhdan group with no nontrivial finite normal subgroups and $F(G, m) \leq_B F(G, n)$, then there exists a G-invariant Borel subset $Z \subseteq (m)^G$ with $\mu_m(Z) = 1$ such that Z admits an n-generator.*

Proof. Suppose that $\varphi : (m)^G \to (n)^G$ is a Borel reduction from $F(G, m)$ to $F(G, n)$. Clearly φ can be extended to a μ_m-nontrivial Borel homomorphism from $E(G, m)$ to $F(G, n)$. Hence, by Theorem 2.3, after slightly adjusting φ if necessary, we can suppose that there exists a G-invariant Borel subset $Z \subseteq (m)^G$ with $\mu_m(Z) = 1$ and an embedding $\pi : G \to G$ such that

$$\varphi(g \cdot z) = \pi(g) \cdot \varphi(z)$$

for all $g \in G$ and $z \in Z$. Since G is co-hopfian, it follows that π is an automorphism of G. Let $\beta = (B_0, B_1, \cdots, B_{n-1})$ be the n-generator of $(n)^G$ given by Example 6.5; and for each $0 \leq i \leq n - 1$, let $A_i = \varphi^{-1}(B_i) \cap Z$. Then we claim that $\alpha = (A_0, A_1, \cdots, A_{n-1})$ is an n-generator of Z. To see this, suppose that $y, z \in Z$ are

distinct points. Clearly if y, z lie in different G-orbits, then $\varphi(y) \neq \varphi(z)$. Otherwise, there exists $1 \neq g \in G$ such that $z = g \cdot y$ and so $\varphi(z) = \pi(g) \cdot \varphi(y) \neq \varphi(y)$. Thus $\varphi(y) \neq \varphi(z)$ and hence there exists $0 \leq i \leq n - 1$ and an element $g \in G$ such that $\varphi(y) \in g \cdot B_i$ and $\varphi(z) \notin g \cdot B_i$. Let $h = \pi^{-1}(g)$. Then $y \in h \cdot A_i$ and $z \notin h \cdot A_i$. Thus $\{ h \cdot A_i \mid 0 \leq i \leq n - 1, h \in G \}$ separates points. $\qquad\square$

Consequently, in order to prove Theorem 6.2, it is enough to rule out the existence of n-generators for full-measure subsets of $(m)^G$ for $n < m$.

Definition 6.7. Suppose that Z is a standard Borel space with Borel probability measure μ and I is a countable set. If $Z = \bigsqcup_{i \in I} A_i$ is a partition into Borel subsets, then the *entropy* of $\alpha = (A_i \mid i \in I)$ is defined to be

$$H(\alpha) = -\sum_{i \in I} \mu(A_i) \log(\mu(A_i)).$$

Remark 6.8. If A_i is a null set, then we define $\mu(A_i) \log(\mu(A_i)) = 0$.

Example 6.9. Suppose that $Z = (m)^G$ and that $\beta = (B_0, B_1, \cdots, B_{m-1})$ is the m-generator given in Example 6.5. Then

$$H(\beta) = -\left(\frac{1}{m} \log\left(\frac{1}{m} \right) + \cdots + \frac{1}{m} \log\left(\frac{1}{m} \right) \right) = \log(m).$$

The following result is well-known. (For example, see Petersen [21, Section 5.1].)

Lemma 6.10. *Suppose that Z is a standard Borel space with Borel probability measure μ. If $\alpha = (A_0, A_1, \cdots, A_{n-1})$ is any partition of Z into Borel subsets, then $H(\alpha) \leq \log(n)$.*

Finally we will make use of the following result, which is implicitly contained in Bowen [2]. Since Bowen does not state this result explicitly, we will briefly explain how to deduce Theorem 6.11 from the results in [2].

Theorem 6.11 (Bowen [2]). *Let $2 \leq m \in \mathbb{N}$. Suppose that G is a countable sofic group and that $Z \subseteq (m)^G$ is a G-invariant Borel subset with $\mu_m(Z) = 1$. If I is a countable set and $\alpha = (A_i \mid i \in I)$ is a finite entropy I-generator of Z, then $H(\alpha) \geq \log(m)$.*

Sketch Proof. Let G be a countable sofic group and let Σ be a sofic approximation of G. Suppose that Z is a standard Borel G-space with G-invariant probability

measure μ. Then for each finite entropy generator $\alpha = (A_i \mid i \in I)$, Bowen defines a corresponding invariant $h(\Sigma, \alpha) \in \mathbb{R} \cup \{ -\infty \}$ with the property that if β is any other finite entropy generator, then $h(\Sigma, \beta) = h(\Sigma, \alpha)$. (For the fact that $h(\Sigma, \alpha)$ does not depend on the choice of α, see Bowen [2, Theorem 2.1].) Furthermore, by Bowen [2, Proposition 5.3], we have that $h(\Sigma, \alpha) \leq H(\alpha)$. (To see this, let $\beta = \{ X \}$ be the trivial partition in the statement of Proposition 5.3.)

Now consider the special case when $Z \subseteq (m)^G$ is a G-invariant Borel subset with $\mu_m(Z) = 1$ and $\alpha = (A_i \mid i \in I)$ is a finite entropy I-generator of Z. Then, by Bowen [2, Theorem 8.1], we have that $h(\Sigma, \alpha) = \log(m)$ and so $H(\alpha) \geq \log(m)$. (The actual statement of Theorem 8.1 refers to (m^G, μ_m), but of course null sets can safely be ignored in this setting.) $\qquad\square$

Proof of Theorem 6.2. If the result fails, then there exist integers $2 \leq n < m$ such that $F(G, m) \leq_B F(G, n)$. Hence, by Lemma 6.6, there exists a G-invariant Borel subset $Z \subseteq (m)^G$ with $\mu_m(Z) = 1$ such that Z admits an n-generator $\alpha = (A_0, A_1, \cdots, A_{n-1})$. Applying Lemma 6.10 and Theorem 6.11, we have that

$$\log(m) \leq H(\alpha) \leq \log(n),$$

which is a contradiction. $\qquad\square$

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