ULTRAPRODUCTS OF FINITE ALTERNATING GROUPS

PAUL ELLIS, SHERWOOD HACHTMAN, SCOTT SCHNEIDER, AND SIMON THOMAS

Abstract. We prove that if $\mathcal{U}$ is a nonprincipal ultrafilter over ω, then the set of normal subgroups of the ultraproduct $\prod_{\mathcal{U}} \mathrm{Alt}(n)$ is linearly ordered by inclusion. We also prove that the number of such ultraproducts up to isomorphism is either $2^{\aleph_0}$ or $2^{2^{\aleph_0}}$, depending on whether or not CH holds.

1. Introduction

If $\mathcal{U}$ is a nonprincipal ultrafilter over ω, then it is easily seen that the ultraproduct $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Alt}(n)$ is not a simple group. However, Elek-Szabó [3] have recently shown that $G_{\mathcal{U}}$ has a unique maximal proper normal subgroup. In this paper, extending their analysis, we shall prove that the set $\mathcal{N}_{\mathcal{U}}$ of normal subgroups of $G_{\mathcal{U}}$ is linearly ordered by inclusion. As we shall see later, this result is an easy consequence of the fact that the set $\mathcal{E}_{\mathcal{U}} = \{ \langle g^{G_{\mathcal{U}}} \rangle \mid 1 \neq g \in G_{\mathcal{U}} \}$ of normal closures of nonidentity elements is linearly ordered by inclusion. More precisely, let $\equiv_{\mathcal{U}}$ be the convex equivalence relation on the linear order $\prod_{\mathcal{U}} \{ 1, \cdots, n \}$ defined by

$$f_{\mathcal{U}} \equiv_{\mathcal{U}} h_{\mathcal{U}} \quad \text{iff} \quad 0 < \lim_{\mathcal{U}} \frac{f(n)}{h(n)} < \infty;$$

and let $L_{\mathcal{U}} = (\prod_{\mathcal{U}} \{ 1, \cdots, n \}) / \equiv_{\mathcal{U}}$, equipped with the quotient linear order. Then we shall prove that $(\mathcal{E}_{\mathcal{U}}, \subset)$ is isomorphic to $L_{\mathcal{U}}$.

In Section 3, we shall compute the number of ultraproducts $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Alt}(n)$ up to isomorphism. Of course, if CH holds, then each such ultraproduct $G_{\mathcal{U}}$ is saturated and hence is determined up to isomorphism by its first order theory; and we shall show that (as expected) there are $2^{\aleph_0}$ many ultraproducts up to elementary equivalence. On the other hand, arguing as in Kramer-Shelah-Tent-Thomas [5], we shall prove that if CH fails, then there exists a family $\{ \mathcal{U}_\alpha \mid \alpha < 2^{2^{\aleph_0}} \}$ of nonprincipal ultrafilters over ω such that the corresponding linear orders $L_{\mathcal{U}_\alpha}$ are

Ellis, Schneider and Thomas were partially supported by NSF Grant DMS 0600940. Hachtman was partially supported by NSF Grant DMS 0603745.

pairwise nonisomorphic. Hence if CH fails, then there are $2^{2^{\aleph_0}}$ many ultraproducts $G_{\mathcal{U}}$ up to isomorphism.

Finally, in Section 4, we shall briefly consider the currently open problems of computing the number of universal sofic groups up to isomorphism and elementary equivalence.

Acknowledgements: After this paper was written, we learned that the results in Section 2 of this paper had been obtained some years earlier by Allsup-Kaye [1].

2. The normal subgroups of $G_{\mathcal{U}}$

Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Alt}(n)$. In this section, we shall prove the following result.

Theorem 2.1. *The collection $\mathcal{N}_{\mathcal{U}}$ of normal subgroups of $G_{\mathcal{U}}$ is linearly ordered by inclusion.*

The following easy observation will enable us to focus our attention on the set $\mathcal{E}_{\mathcal{U}} = \{ \langle g^{G_{\mathcal{U}}} \rangle \mid 1 \neq g \in G_{\mathcal{U}} \}$ of normal closures of nonidentity elements.

Lemma 2.2. *If G is any group, then the following statements are equivalent.*

(a) *The set of normal subgroups of G is linearly ordered by inclusion.*
(b) *The set of normal closures of nonidentity elements of G is linearly ordered by inclusion.*

Proof. Clearly (a) implies (b). Conversely, assume that (b) holds and let N, M be normal subgroups of G. If for every $g \in N$, there exists $h \in M$ such that $g \in \langle h^G \rangle$, then clearly $N \leqslant M$. Otherwise, there exists $g \in N$ such that for every $h \in M$, we have that $\langle g^G \rangle \nleqslant \langle h^G \rangle$ and so $\langle h^G \rangle \leqslant \langle g^G \rangle$, which implies that $M \leqslant N$. □

For each $\pi \in \mathrm{Alt}(n)$, let $\mathrm{supp}(\pi) = \{\ell \mid \pi(\ell) \neq \ell\}$. In [3], Elek-Szabó proved that if $g = (\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}}$, then

$$\langle g^{G_{\mathcal{U}}} \rangle = G_{\mathcal{U}} \qquad \text{iff} \qquad \lim_{\mathcal{U}} \frac{|\,\mathrm{supp}(\pi_n)|}{n} > 0.$$

(This is an immediate consequence of Elek-Szabó [3, Proposition 2.3].) It follows that $M_{\mathcal{U}} = \{ (\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \mid \lim_{\mathcal{U}} \frac{|\,\mathrm{supp}(\pi_n)|}{n} = 0 \}$ is the unique maximal proper normal subgroup of $G_{\mathcal{U}}$. This suggests that, in order to understand the normal closure of an element $(\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}}$, we should consider the relative growth rate of

$|\operatorname{supp}(\pi_n)|$. From now on, we adopt the convention that if $(\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \smallsetminus 1$, then we always choose (π_n) such that $\pi_n \neq 1$ for all $n \in \omega$; and the normal closure of $(\pi_n)_{\mathcal{U}}$ will be denoted by $N_{(\pi_n)_{\mathcal{U}}}$.

Definition 2.3. Let $\preceq$ be the quasi-order on $G_{\mathcal{U}} \smallsetminus 1$ defined by

$$(\pi_n)_{\mathcal{U}} \preceq (\varphi_n)_{\mathcal{U}} \quad \text{iff} \quad \lim_{\mathcal{U}} \frac{|\operatorname{supp}(\pi_n)|}{|\operatorname{supp}(\varphi_n)|} < \infty.$$

Proposition 2.4. *If $(\pi_n)_{\mathcal{U}}$, $(\varphi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \smallsetminus 1$ are nonidentity elements, then*

$$(\pi_n)_{\mathcal{U}} \in N_{(\varphi_n)_{\mathcal{U}}} \iff (\pi_n)_{\mathcal{U}} \preceq (\varphi_n)_{\mathcal{U}}.$$

We shall split the proof of Proposition 2.4 into a sequence of lemmas. We shall begin by proving the easier implication.

Lemma 2.5. *If $(\pi_n)_{\mathcal{U}}$, $(\varphi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \smallsetminus 1$ and $(\pi_n)_{\mathcal{U}} \in N_{(\varphi_n)_{\mathcal{U}}}$, then $(\pi_n)_{\mathcal{U}} \preceq (\varphi_n)_{\mathcal{U}}$.*

Proof. If $(\pi_n)_{\mathcal{U}} \in N_{(\varphi_n)_{\mathcal{U}}}$, then there exists an integer $k \geq 1$ such that $(\pi_n)_{\mathcal{U}}$ can be expressed as a product of k conjugates of $(\varphi_n)_{\mathcal{U}}^{\pm 1}$. Hence for $\mathcal{U}$-a.e. $n \in \mathbb{N}$, the permutation π_n can be expressed as a product of k conjugates of $\varphi_n^{\pm 1}$. This implies that $|\operatorname{supp}(\pi_n)| \leq k|\operatorname{supp}(\varphi_n)|$ and so $\lim_{\mathcal{U}} |\operatorname{supp}(\pi_n)|/|\operatorname{supp}(\varphi_n)| \leq k$. □

Recall that a permutation $\sigma \in \operatorname{Alt}(m)$ is said to be *exceptional* iff its conjugacy class $\sigma^{\operatorname{Sym}(m)}$ splits into two conjugacy classes in $\operatorname{Alt}(m)$. It is well-known that this occurs iff the cycles of σ have distinct odd lengths.

Lemma 2.6. *If $\sigma \in \operatorname{Alt}(m)$ is a nonexceptional fixed-point-free permutation, then every element of $\operatorname{Alt}(m)$ can be expressed as a product of exactly 4 conjugates of σ.*

Proof. This is an immediate consequence of Brenner [2, Theorem 3.05]. □

Lemma 2.7. *If $(\pi_n)_{\mathcal{U}}$, $(\varphi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \smallsetminus 1$ and $(\pi_n)_{\mathcal{U}} \preceq (\varphi_n)_{\mathcal{U}}$, then $(\pi_n)_{\mathcal{U}} \in N_{(\varphi_n)_{\mathcal{U}}}$.*

Proof. Suppose that $(\pi_n)_{\mathcal{U}} \preceq (\varphi_n)_{\mathcal{U}}$. As mentioned earlier, Elek-Szabó [3] have proved that if $\lim_{\mathcal{U}} \frac{|\operatorname{supp}(\varphi_n)|}{n} > 0$, then $N_{(\varphi_n)_{\mathcal{U}}} = G_{\mathcal{U}}$. Hence we can suppose that $\lim_{\mathcal{U}} \frac{|\operatorname{supp}(\varphi_n)|}{n} = 0$. Let $\lim_{\mathcal{U}} |\operatorname{supp}(\pi_n)|/|\operatorname{supp}(\varphi_n)| \leq k$, where $k \geq 2$ is an integer. Then for $\mathcal{U}$-a.e. $n \in \mathbb{N}$, we have that $|\operatorname{supp}(\pi_n)| \leq k|\operatorname{supp}(\varphi_n)| \leq n$. Hence there exists a permutation $\sigma_n \in \operatorname{Alt}(n)$ such that the following conditions are satisfied:

(a) σ_n is a product of k conjugates $\psi_1, \cdots, \psi_k$ of φ_n.

(b) If $1 \leq i < j \leq k$, then $\mathrm{supp}(\psi_i) \cap \mathrm{supp}(\psi_j) = \emptyset$.

(c) $\mathrm{supp}(\pi_n) \subseteq \mathrm{supp}(\sigma_n)$.

Regarding σ_n as an element of $\mathrm{Alt}(\mathrm{supp}(\sigma_n))$, we see that σ_n is a nonexceptional fixed-point-free permutation. Hence, applying Lemma 2.6, it follows that π_n is a product of 4 conjugates of σ_n and this implies that $(\pi_n)_{\mathcal{U}}$ is a product of $4k$ conjugates of $(\varphi_n)_{\mathcal{U}}$. $\square$

Applying Proposition 2.4, it follows that if $(\pi_n)_{\mathcal{U}}, (\varphi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \smallsetminus 1$ are nonidentity elements, then

$$N_{(\pi_n)_{\mathcal{U}}} = N_{(\varphi_n)_{\mathcal{U}}} \qquad \text{iff} \qquad 0 < \lim_{\mathcal{U}} \frac{|\,\mathrm{supp}(\pi_n)|}{|\,\mathrm{supp}(\varphi_n)|} < \infty;$$

and that $(\,\mathcal{E}_{\mathcal{U}}, \subset\,)$ is isomorphic to the linear order $L_{\mathcal{U}} = (\,\prod_{\mathcal{U}}\{\,1, \cdots, n\,\}\,)/{\equiv_{\mathcal{U}}}$. This completes the proof of Theorem 2.1.

Remark 2.8. Clearly $L_{\mathcal{U}}$ has a least element; namely, the $\equiv_{\mathcal{U}}$-class containing the constant functions. If we identify $G_{\mathcal{U}}$ with its image under the embedding

$$G_{\mathcal{U}} \to \mathrm{Sym}(\prod_{\mathcal{U}}\{\,1, \cdots, n\,\})$$

corresponding to the natural action

$$(\pi_n)_{\mathcal{U}} \cdot (\ell_n)_{\mathcal{U}} = (\pi_n(\ell_n))_{\mathcal{U}}$$

of $G_{\mathcal{U}}$ on $\prod_{\mathcal{U}}\{\,1, \cdots, n\,\}$, then the minimal nontrivial normal subgroup of $G_{\mathcal{U}}$ is the group $\mathrm{Alt}(\prod_{\mathcal{U}}\{\,1, \cdots, n\,\})$ of finite even permutations of $\prod_{\mathcal{U}}\{\,1, \cdots, n\,\}$. Hence, by Scott [7, 11.4.7], since

$$\mathrm{Alt}(\prod_{\mathcal{U}}\{\,1, \cdots, n\,\}) \leqslant G_{\mathcal{U}} \leqslant \mathrm{Sym}(\prod_{\mathcal{U}}\{\,1, \cdots, n\,\}),$$

it follows that $\mathrm{Aut}(G_{\mathcal{U}})$ is precisely the normalizer of $G_{\mathcal{U}}$ in $\mathrm{Sym}(\prod_{\mathcal{U}}\{\,1, \cdots, n\,\})$. Of course, if CH holds, then the ultraproduct $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Alt}(n)$ is saturated and so $|\,\mathrm{Aut}(G_{\mathcal{U}})| = 2^{\aleph_1}$.

Question 2.9. Is it consistent that $\mathrm{Aut}(G_{\mathcal{U}}) = \prod_{\mathcal{U}} \mathrm{Sym}(n)$?

3. The number of nonisomorphic ultraproducts

In this section, we shall compute the number of ultraproducts $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Alt}(n)$ up to isomorphism. If CH holds, then each such ultraproduct is saturated and hence is determined up to isomorphism by its first order theory. So the following result implies that if CH holds, then there exist exactly $2^{\aleph_0}$ ultraproducts $\prod_{\mathcal{U}} \mathrm{Alt}(n)$ up to isomorphism.

Theorem 3.1. *There exist $2^{\aleph_0}$ many ultraproducts $\prod_{\mathcal{U}} \mathrm{Alt}(n)$ up to elementary equivalence.*

Proof. For each prime $p \geq 5$, let $D_p = \{n \in \omega \mid n \geq p \text{ and } n \equiv 0, 1, 2 \mod p\}$.

Claim 3.2. *There exists a first order sentence Φ_p such that for all $n \geq 7$,*

$$n \in D_p \qquad \textit{iff} \qquad \mathrm{Alt}(n) \vDash \Phi_p.$$

Proof of Claim 3.2. Clearly $n \in D_p$ iff $\mathrm{Alt}(n)$ contains an element of order p with at most 2 fixed points. To see that this property is first order definable, first note that an element $\pi \in \mathrm{Alt}(n)$ of order $p \geq 5$ has at most 2 fixed points iff there does not exist a 3-cycle $\sigma \in \mathrm{Alt}(n)$ which commutes with π. Also note that if $n \geq 7$, then an element $\sigma \in \mathrm{Alt}(n)$ of order 3 is a 3-cycle iff $\sigma\psi\sigma\psi^{-1}$ has order at most 5 for all $\psi \in \mathrm{Alt}(n)$. □

Let $\mathbb{P} = \{\, p \in \omega \mid p \geq 5 \text{ is prime} \,\}$. Then it is enough to check that for each subset $S \subseteq \mathbb{P}$, the collection $\mathcal{D}_S = \{\, D_p \mid p \in S \,\} \cup \{\, \omega \smallsetminus D_p \mid p \in \mathbb{P} \smallsetminus S \,\}$ has the finite intersection property. So suppose that $p_1, \cdots, p_\ell \in S$ and that $q_1, \cdots, q_m \in \mathbb{P} \smallsetminus S$. By the Chinese Remainder Theorem, there exists a positive integer $n \in \omega$ such that

- $n \equiv 0 \mod p_i$ for all $1 \leq i \leq \ell$; and
- $n \equiv 3 \mod q_j$ for all $1 \leq j \leq m$.

Clearly $n \in D_{p_1} \cap \cdots \cap D_{p_\ell} \cap (\omega \smallsetminus D_{q_1}) \cap \cdots \cap (\omega \smallsetminus D_{q_m})$. □

In order to compute the number of nonisomorphic ultraproducts $G_{\mathcal{U}}$ when CH fails, we shall focus our attention on the linearly ordered set $(\,\mathcal{E}_{\mathcal{U}}, \subset\,)$ of normal closures of nonidentity elements. Clearly if $\mathcal{U}$, $\mathcal{B}$ are nonprincipal ultrafilters over ω and $G_{\mathcal{U}} \cong G_{\mathcal{B}}$, then $(\,\mathcal{E}_{\mathcal{U}}, \subset\,) \cong (\,\mathcal{E}_{\mathcal{B}}, \subset\,)$. Furthermore, in Section 2, we showed that $(\,\mathcal{E}_{\mathcal{U}}, \subset\,)$ is isomorphic to $L_{\mathcal{U}} = (\prod_{\mathcal{U}}\{\,1, \cdots, n\,\})/{\equiv_{\mathcal{U}}}$ and clearly $L_{\mathcal{U}}$ can be regarded as an initial segment of $(\prod_{\mathcal{U}} \omega)/{\equiv_{\mathcal{U}}}$. Hence the following result implies if CH fails, then there exist $2^{2^{\aleph_0}}$ ultraproducts $G_{\mathcal{U}}$ up to isomorphism.

Definition 3.3. If L_1, L_2 are linear orders, then $L_1 \approx_i^* L_2$ iff L_1 and L_2 have nonempty isomorphic initial segments I_1, I_2 with $|I_1|$, $|I_2| > 1$.

The requirement that $|I_1|$, $|I_2| > 1$ is needed in Definition 3.3 because of the fact that each linear order $L_{\mathcal{U}} = (\prod_{\mathcal{U}}\{1, \cdots, n\})/{\equiv_{\mathcal{U}}}$ has a first element; namely, the $\equiv_{\mathcal{U}}$-class containing the constant functions.

Theorem 3.4. *If CH fails, then there exists a set $\{\mathcal{U}_\alpha \mid \alpha < 2^{2^{\aleph_0}}\}$ of nonprincipal ultrafilters over ω such that*

$$(\prod_{\mathcal{U}_\alpha} \omega)/{\equiv_{\mathcal{U}_\alpha}} \;\not\approx_i^*\; (\prod_{\mathcal{U}_\beta} \omega)/{\equiv_{\mathcal{U}_\beta}}$$

for all $\alpha < \beta < 2^{2^{\aleph_0}}$.

Proof. The proof of Kramer-Shelah-Tent-Thomas [5, Theorem 3.3] goes through with just one minor change; namely, in the proof of Lemma 4.7, the collection $\{B_{s,t} \mid s < t \in I\}$, where $B_{s,t} = \{n \in \omega \mid f_s(n) < f_t(n)\}$, is replaced by $\{B_{s,t,k} \mid s < t \in I \text{ and } 1 \leq k \in \omega\}$, where $B_{s,t,k} = \{n \in \omega \mid k f_s(n) < f_t(n)\}$. □

4. Universal sofic groups

In this final section, we shall briefly consider the currently open problems of computing the number of the universal sofic groups up to isomorphism and elementary equivalence.

Recall that if $\mathcal{U}$ is a nonprincipal ultrafilter over ω and $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Alt}(n)$, then

$$M_{\mathcal{U}} = \left\{ (\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}} \mid \lim_{\mathcal{U}} \frac{|\operatorname{supp}(\pi_n)|}{n} = 0 \right\}$$

is the unique maximal proper normal subgroup of $G_{\mathcal{U}}$. Let $S_{\mathcal{U}} = G_{\mathcal{U}}/M_{\mathcal{U}}$. Then by Elek-Szabó [3], if Γ is a finitely generated group, then the following statements are equivalent:

- Γ is a sofic group.
- Γ embeds into $S_{\mathcal{U}}$ for some (equivalently every) nonprincipal ultrafilter $\mathcal{U}$.

For this reason, $S_{\mathcal{U}}$ is said to be a *universal sofic group.* (A clear account of the basic theory of sofic groups can be found in Pestov [6]. It is an important open problem whether every finitely generated group is sofic.)

It is natural to conjecture that the number of universal sofic groups up to isomorphism is either $2^{\aleph_0}$ or $2^{2^{\aleph_0}}$, depending on whether or not CH holds. However, it is currently not even known whether it is consistent that there exist two nonisomorphic universal sofic groups.

Question 4.1. Compute the number of universal sofic groups up to isomorphism.

In Section 3, simple arithmetic considerations enabled us to construct $2^{\aleph_0}$ nonelementarily equivalent ultraproducts $\prod_{\mathcal{U}} \mathrm{Alt}(n)$. However, factoring by the maximal proper normal subgroup $M_{\mathcal{U}}$ appears to eliminate all the arithmetic aspects of the group $S_{\mathcal{U}}$. For example, Glebsky-Rivera [4] have recently shown that if $g \in S_{\mathcal{U}}$ and if p is any prime, then there exists $h \in S_{\mathcal{U}}$ such that $h^p = g$.

Question 4.2. Are all universal sofic groups $S_{\mathcal{U}}$ elementarily equivalent?

References

[1] J. Allsup and R. Kaye, *Normal subgroups of nonstandard symmetric and alternating groups*, Arch. Math. Logic **46** (2007), 107–121.

[2] J. L. Brenner, *Covering theorems for FINASIGs. VIII. Almost all conjugacy classes in $\mathcal{A}_n$ have exponent* ≤ 4. J. Austral. Math. Soc. Ser. A **25** (1978), 210–214.

[3] G. Elek and E. Szabó, *Hyperlinearity, essentially free actions and L^2-invariants. The sofic property.* Math. Ann. **332** (2005), 421–441.

[4] L. Glebsky and L. M. Rivera, *Almost solutions of equations in permutations*, preprint, 2008.

[5] L. Kramer, S. Shelah, K. Tent and S. Thomas *Asymptotic cones of finitely presented groups*, Adv. Math. **193** (2005), 142–173.

[6] V. Pestov, *Hyperlinear and sofic groups: a brief guide*, to appear in Bull. Symbolic Logic.

[7] W. R. Scott, *Group theory*, Dover, New York, 1987.

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA

E-mail address: prellis@math.rutgers.edu

E-mail address: hachtman@eden.rutgers.edu

E-mail address: scottsch@math.rutgers.edu

E-mail address: sthomas@math.rutgers.edu