Solutions to Homework 7

Problem #1 (3 points)

Reflexive:
$(a_1, a_2)R(a_1, a_2)$ since E_1 and E_2 are themselves reflexive and therefore $a_1E_1a_1$ and $a_2E_2a_2$.

Symmetric:

$$\begin{aligned}(a_1, a_2)R(b_1, b_2) &\Leftrightarrow a_1E_1b_1 \textit{ and } a_2E_2b_2\\ &\Leftrightarrow b_1E_1a_1 \textit{ and } b_2E_2a_2 (\text{since } E_1 \text{ and } E_2 \text{ are themselves symmetric})\\ &\Leftrightarrow (b_1, b_2)R(a_1, a_2)\end{aligned}$$

Transitive:
Assume that $(a_1, a_2)R(b_1, b_2$ and $(a_1, a_2)R(c_1, c_2)$. This is true if and only if:

$$\begin{aligned}&a_1E_1b_1 \textit{ and } a_2E_2b_2 \textit{ and } a_1E_1c_1 \textit{ and } a_2E_2c_2\\ &\Leftrightarrow b_1E_1c_1 \textit{ and } b_2E_2c_2 (\text{by the transitivity of } E_1 \text{ and } E_2)\\ &\Leftrightarrow (b_1, b_2)R(c_1, c_2)\end{aligned}$$

Therefore, R is an equivalence relation.

Problem #2 (3 points)

This is not a partial ordering since it is not antisymmetric. Consider $(a, b) = (1, 4)$ and $(c, d) = (2, 3)$. Then, $(a, b)R(c, d)$ since $1 \leq 2$. But also, $(c, d)R(a, b)$ since $3 \leq 4$. But, $(a, b) \neq (c, d)$.

It can also be easily shown that this is not transitive. Consider $(a, b) = (4, 6)$, $(c, d) = (5, 2)$, and $(e, f) = (1, 3)$. Then we have $(a, b)R(c, d)$, $(c, d)R(e, f)$, but we do NOT have that $(a, b)R(e, f)$.

Problem #3a (2 points)

An immediate successor of an element a is one such that $a|b$ and there does not exist $c \neq b, c \neq a$ such that $a|c$ and $c|b$.

If $b = pa$, p prime, then b is an immediate successor. To see this, consider c such that $a|c$ and $c|pa$.

Since $a|c$, then we have that $c = ja$. Now since $ja|pa$, we see that $j|p$. But the only divisors of p are 1 and p. If $j = 1$, then $ja = a$, a contradiction. And if $j = p$, then $ja = pa$, a contradiction.

So, pa is an immediate successor of a for each prime p. Since there are infinitely many primes, we have infinitely many immediate successors.

Problem #3b (2 points)

We want to show that K is bounded above $\Leftrightarrow$ K is finite.

"$\Rightarrow$"
K bounded above means that there exists b such that $k|b$ for every $k \in K$. By the nature of the expression $k|b$, this implies that $k \leq b$ (you can't be something's divisor and be LARGER). Therefore, $k \leq b$ for all $k \in K$. Since there are finitely many numbers less than any natural number b (namely b many of them), K must be finite.

"$\Leftarrow$"
K is finite, so list out its elements as $k_1, k_2, ..., k_m$. Then certainly $b = k_1 k_2 ... k_m$ is an upper bound for K, so K is bounded above.