Solutions to Homework 6

Problem #1 (2 points)

Choose $(x, y) \in A \times (B \cap C)$ arbitrarily. Then we have:

$$\begin{aligned}
(x, y) \in A \times (B \cap C) &\Leftrightarrow x \in A \textit{ and } y \in B \cap C \\
&\Leftrightarrow x \in A \textit{ and } (y \in B \textit{ and } y \in C) \\
&\Leftrightarrow (x \in A \textit{ and } y \in B) \textit{ and } (x \in A \textit{ and } y \in C) \\
&\Leftrightarrow (x, y) \in A \times B \textit{ and } (x, y) \in A \times C \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)
\end{aligned}$$

So we have our desired result, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Problem #2a (2 points)

Choose $(x, y) \in (A \times B) \cup (C \times D)$ arbitrarily. Then we have:

$$\begin{aligned}
(x, y) \in (A \times B) \cup (C \times D) &\Rightarrow (x, y) \in (A \times B) \textit{ or } (x, y) \in (C \times D) \\
&\Rightarrow (x \in A \textit{ and } y \in B) \textit{ or } (x \in C \textit{ and } y \in D) \\
* &\Rightarrow (x \in A \textit{ or } x \in C) \textit{ and } (y \in B \textit{ or } y \in D) \\
&\Rightarrow (x \in A \cup C) \textit{ and } (y \in B \cup D) \\
&\Rightarrow (x, y) \in (A \cup C) \times (B \cup D)
\end{aligned}$$

*(Note for part b, this is the step that we have failed if we tried to use $\Leftrightarrow$'s)

So, we have our desired results, $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Problem #2b (1 point)

From the hint in part a, it's semi-obvious that we want to choose a set such that we have a unique $x \in A$ and a unique $y \in D$ (or equivalently C and B). So, the simplest such example would be $A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$.

Then $(A \times B) \cup (C \times D) = \{(1, 2), (3, 4)\}$, where as $(A \cup C) \times (B \cup D) = \{(1, 2), (1, 4), (3, 2), (3, 4)\}$.

Problem #3 (2 points)

(i) Reflexive: $(a,b) \sim (a,b)$ because $ab = ba$.
(ii) Symmetric: $(a,b) \sim (c,d) \Leftrightarrow ad = bc \Leftrightarrow cb = da \Leftrightarrow (c,d) \sim (a,b)$.
(iii) Transitive:

$$\begin{aligned}(a,b) \sim & (c,d) \text{ and } (a,b) \sim (e,f) \\ & \Rightarrow ad = bc \text{ and } af = be \\ & \Rightarrow \frac{a}{b} = \frac{c}{d} \text{ and } \frac{a}{b} = \frac{e}{f} \\ & (\text{remember } b,d,f \neq 0 \text{ by the definition of } S) \\ & \Rightarrow \frac{c}{d} = \frac{e}{f} \\ & \Rightarrow cf = de \\ & \Rightarrow (c,d) \sim (e,f).\end{aligned}$$

Therefore, $\sim$ is an equivalence relation.

Problem #4 (3 points)

(i) Reflexive: aRa because $\frac{a}{a} = 1 \in \mathbb{Q}$ (also $a - a = 0 \in \mathbb{Q}$).
(ii) Symmetric: $aRb \Leftrightarrow \frac{a}{b} \in \mathbb{Q}$ or $a - b \in \mathbb{Q}$.
But, we know that $\frac{a}{b} \in \mathbb{Q} \Leftrightarrow \frac{b}{a} \in \mathbb{Q}$. Also, $a - b \in \mathbb{Q} \Leftrightarrow b - a \in \mathbb{Q}$. So, $aRb \Leftrightarrow bRa$.
(Justification: If $\frac{a}{b} = \frac{p}{q}$, $p, q \in \mathbb{Z} \setminus \{0\}$, then, $\frac{b}{a} = \frac{q}{p}$. And, if $a - b = \frac{p}{q}$, then, $b - a = \frac{-p}{q}$)
(iii)Transitive: This is where we run into problems and we can show that this is actual NOT an equivalence relation. But, it's "close".

Consider aRb and aRc. If $\frac{a}{b} = \frac{p}{q} \in \mathbb{Q}$ and $\frac{a}{c} = \frac{s}{t} \in \mathbb{Q}$, it is easy to see that $\frac{b}{c} \in \mathbb{Q}$. Because $\frac{b}{c} = \frac{ab}{ac} = \frac{a}{c}\frac{b}{a} = \frac{sq}{pt} \in \mathbb{Q}$

Also if $a - b = \frac{p}{q} \in \mathbb{Q}$ and $a - c = \frac{s}{t} \in \mathbb{Q}$, then, $b - c = (b-a) + (a-c) = \frac{-p}{q} + \frac{s}{t} = \frac{-pt+qs}{qt} \in \mathbb{Q}$.

So, in these two special cases we do have that (aRb and aRc) $\Rightarrow bRc$. The problem arises in the third case.

Consider $a = \sqrt{2}, b = 2\sqrt{2}, c = 1 + \sqrt{2}$. Then, $\frac{a}{b} = \frac{1}{2}$, so aRb, and $a - c = \sqrt{2} - (1 + \sqrt{2}) = -1$, so aRc.

But, $\frac{c}{b} = \frac{1+\sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2}+2}{4} = \frac{\sqrt{2}}{4} + \frac{1}{2}$. But this is not rational since (1)an irrational times a non-zero rational is irrational and (2)an irrational plus a rational is irrational.

And, $b - c = 2\sqrt{2} - (1 + \sqrt{2}) = \sqrt{2} - 1$, which is not rational by property (2) above. So, in this case aRb and aRc, but not bRc. So, transitivity fails and so this is NOT an equivalence relation.