Solutions to Homework 5

Problem #1 (2 points)

Base case, n = 1:

$$\sum_{i=1}^{1} i^3 = 1$$
$$\left[\frac{1(1+1)}{2}\right]^2 = 1 \ \checkmark$$

Assume, the equation holds up to some n, then:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2+4(n+1))}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left[\frac{(n+1)(n+2)}{2}\right]^2
\end{aligned}$$

Thus showing by induction that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$, $\forall n \in \mathbb{N}$.

Problem #2 (2 points)

Base case, n = 1:

$$a_1 = \frac{9^1 - 5^1}{2} = \frac{4}{2} = 2 \ \checkmark$$

Assume the equation holds up to some n, then:

$$\begin{aligned} a_{n+1} &= 7a_n + 9^n + 5^n \\ &= \frac{7(9^n - 5^n)}{2} + 9^n + 5^n \\ &= \frac{7 \cdot 9^n + 2 \cdot 9^n - 7 \cdot 5^n + 2 \cdot 5^n}{2} \\ &= \frac{9 \cdot 9^n - 5 \cdot 5^n}{2} \\ &= \frac{9^{n+1} - 5^{n+1}}{2} \end{aligned}$$

Thus showing by induction that $a_n = \frac{9^n - 5^n}{2}$, $\forall n \in \mathbb{N}$, when defined by this recursion.

Problem #3 (3 points)

Base case, n = 1, n + 1 = 2:

$$\begin{aligned} a_1 &= 2^1 - 1 = 1 \checkmark \\ a_2 &= 2^2 - 1 = 3 \checkmark \end{aligned}$$

Assume the equation holds up to some n, n+1, then:

$$\begin{aligned} a_{n+2} &= 3a_{n+1} - 2a_n \\ &= 3(2^{n+1} - 1) - 2(2^n - 1) \\ &= 3 \cdot 2^{n+1} - 3 - 2 \cdot 2^n + 2 \\ &= 3 \cdot 2^{n+1} - 2^{n+1} - 1 \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

Thus showing by induction that $a_n = 2^n - 1$, $\forall n \in \mathbb{N}$, when defined by this recursion.

Problem #4 (3 points)

Base case, n = 1, a 2×2 square grid with a square removed can obviously be covered by an L-shaped tile.

Assume that we can cover a $2^n \times 2^n$ square grid. Then, for a $2^{n+1} \times 2^{n+1}$ square grid, we can divide it into 4 quadrants, all of size $2^n \times 2^n$. Now the tile we remove must fall into one of these quadrants, so by our assumption we can cover this quadrant. For the remaining 3 quadrants, we can place an L-shaped tile in the middle that straddles all 3 remaining quadrants. Now, covering these quadrants is identical to covering a $2^n \times 2^n$ square grid with one square removed, which again we know we can do by our assumption.

Thus, we have shown by induction that for any $n \in \mathbb{N}$, we can cover a $2^n \times 2^n$ square grid with one square removed by L-shaped tiles.