Solutions to Homework 4

Problem #1 (2 points)

Base case, n = 1:

$$\sum_{i=1}^{1} i^2 = 1$$
$$\frac{1(1+1)(2(1)+1)}{6} = 1 \ \surd$$

Assume the equation holds up to some n, then:

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= (n+1)^2 + \sum_{i=1}^{n} i^2 \\ &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n^2+n+6n+6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

Thus showing by induction that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall\, n \in \mathbb{N}$.

Problem #2 (3 points)

Base case, n = 1 (Note, that it varies by who you ask whether or not 0 is a natural number. The book defines $\mathbb{N}$ as NOT containing 0 (p. 96) , so I followed this convention. However, I did not penalize if people chose 0 as a base case here):

$$1 + x^1 = x + 1$$
$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1 \ \surd$$

Assume the equation holds up to some n, then:

$$\begin{aligned}1 + x + x^2 + ... + x^n + x^{n+1} &= \frac{x^{n+1}-1}{x-1} + x^{n+1} \\ &= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x-1} \\ &= \frac{x^{n+2}-1}{x-1}\end{aligned}$$

Thus showing by induction that $1 + x + ... + x^n = \frac{x^{n+1}-1}{x-1}$.

Problem #3 (2 points)

Base case, n = 1:

$$\sum_{i=1}^{1} \frac{1}{(i+2)(i+3)} = \frac{1}{12}$$
$$\frac{1}{3} - \frac{1}{1+3} = \frac{1}{12} \checkmark$$

Assume the equation holds up to some n, then:

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{(i+2)(i+3)} &= \frac{1}{(n+3)(n+4)} + \sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} \\
&= \frac{1}{(n+3)(n+4)} + \frac{1}{3} - \frac{1}{n+3} \\
&= \frac{1}{3} + \frac{1-(n+4)}{(n+3)(n+4)} \\
&= \frac{1}{3} + \frac{-n-3}{(n+3)(n+4)} \\
&= \frac{1}{3} - \frac{1}{n+4}
\end{aligned}$$

Thus showing by induction that $\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} = \frac{1}{3} - \frac{1}{n+3}$.

Problem #4 (3 points)

Base case, 1 set:

$$A_1 \subseteq A_1 \checkmark$$

Assume the statement holds up to some n, then, for sets $A_1, A_2, ..., A_n, A_{n+1}$:

$$\begin{aligned}
&\textit{There exists } k,\ 1 \leq k \leq n,\ \textit{such that } A_k \subseteq A_i\ \forall\ i,\ 1 \leq i \leq n. \\
&\textit{Now, for the set } A_{n+1}\textit{, there are two distinct possibilities}: \\
&\quad (1) A_k \subseteq A_{n+1},\ \textit{in which case } A_k \subseteq A_i\ \forall\ i, 1 \leq i \leq \mathbf{n+1}\ \textit{or} \\
&\quad (2) A_{n+1} \subseteq A_k\ (\textit{and obviously } A_{n+1} \subseteq A_{n+1}),\ \textit{in which case } A_{n+1} \subseteq A_i\ \forall\ i,\ 1 \leq i \leq n+1
\end{aligned}$$

Problem #2.4.13, Towers of Hanoi (3 points)

Base case, 1 disk:

$$\begin{aligned}
&\textit{Obviously it only takes } 1 \textit{ move to transfer } 1 \textit{ disk, and} \\
&2^1 - 1 = 1 \checkmark
\end{aligned}$$

Assume the solution for the puzzle is true up to n disks, then for $n+1$ disks:

$$\begin{aligned}
&\textit{It takes } 2^n - 1 \textit{ moves to transfer the top } n \textit{ disks (by induction)} \\
&\textit{It takes } 1 \textit{ move to transfer the } (n+1)^{st} \textit{ (largest) disk to the remaining open peg} \\
&\textit{It takes another } 2^n - 1 \textit{ moves to transfer again the top } n \textit{ disks to the same peg as the largest}
\end{aligned}$$

So, the entire sequence takes $(2^n - 1) + 1 + (2^n - 1) = 2 * 2^n - 1 = 2^{n+1} - 1$ moves. Thus showing by induction that n disks can be transferred in $2^n - 1$ moves.