Solutions to Homework 3

Problem #1 Choose $x \in (A \cup B) \backslash (A \cap B)$ arbitrarily, then:

$$\begin{aligned}
x \in (A \cup B) \backslash (A \cap B) &\Leftrightarrow (x \in A \cup B) \textit{ and not } (x \in A \cap B) \\
&\Leftrightarrow (x \in A \textit{ or } x \in B) \textit{ and not } (x \in A \textit{ and } x \in B) \\
&\Leftrightarrow (x \in A \textit{ or } x \in B) \textit{ and } (x \notin A \textit{ or } x \notin B) \\
&\Leftrightarrow (x \in A \textit{ and } x \notin A) \textit{ or } (x \in A \textit{ and } x \notin B) \textit{ or } (x \in B \textit{ and } x \notin A) \textit{ or } (x \in B \textit{ and } x \notin B) \\
&\qquad \text{Note that we cannot have } (x \in A \textit{ and } x \notin A) \text{ and likewise for B} \\
&\Leftrightarrow (x \in A \textit{ and } x \notin B) \textit{ or } (x \in B \textit{ and } x \notin A) \\
&\Leftrightarrow (x \in A \backslash B) \textit{ or } (x \in B \backslash A) \\
&\Leftrightarrow x \in (A \backslash B) \cup (B \backslash A)
\end{aligned}$$

Since our choice of x was arbitary, this gives us $(A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$

Problem #2a This is false, consider the following counterexample:

$$\begin{aligned}
&A = \{1\},\ B = \{2\},\ C = \{3\}. \\
&Then,\ A \cup (B \backslash C) = \{1,\ 2\} \\
&And,\ (A \cup B) \backslash (A \cup C) = \{2\}.
\end{aligned}$$

Problem #2b This is true by the following proof. Choose $x \in A \cap (B \backslash C)$ arbitrarily, then:

$$\begin{aligned}
x \in A \cap (B \backslash C) &\Leftrightarrow (x \in A) \textit{ and } (x \in B \textit{ and } x \notin C) \\
&\Leftrightarrow x \in A \textit{ and } x \in B \textit{ and } x \notin C \\
&\Leftrightarrow (x \in A \textit{ and } x \in B) \textit{ and } (x \notin C \textit{ or } x \notin A) \\
&\qquad \text{Note that we can add } x \notin A \text{ since we know this must always be false.} \\
&\Leftrightarrow (x \in A \cap B) \textit{ and not } (x \in C \textit{ and } x \in A) \\
&\Leftrightarrow (x \in A \cap B) \textit{ and not } (x \in A \cap C) \\
&\Leftrightarrow x \in (A \cap B) \backslash (A \cap C)
\end{aligned}$$

Since our choice of x was arbitrary, this gives us $A \cap (B \backslash C) = (A \cap B) \backslash (A \cap C)$.

Problem #3a

$$\bigcup_{q \in \mathbb{Q}^+} D_q = \mathbb{R} \textit{ since } (\frac{1}{2} - q,\ \frac{1}{2} + q) \to (-\infty,\ +\infty) \textit{ as } q \to +\infty$$

$$\bigcap_{q \in \mathbb{Q}^+} D_q = \{\frac{1}{2}\} \textit{ since } \frac{1}{2} \in (\frac{1}{2} - q,\ \frac{1}{2} + q)\ \forall q \in \mathbb{Q}^+$$

Note that $0 \notin \mathbb{Q}^+$.

Problem #3b

$$\bigcup_{q\in\mathbb{Q}} K_q = \mathbb{R} \textit{ since clearly } (\mathbb{R}\backslash\{q_1\}) \cup (\mathbb{R}\backslash\{q_2\}) = \mathbb{R}\ \forall q_1, q_2 \in \mathbb{Q}$$

$$\bigcap_{q\in\mathbb{Q}} K_q = \mathbb{R}\backslash\mathbb{Q} \textit{ since each } q \in \mathbb{Q} \textit{ is missing from exactly one of the } K_q\ 's$$

Problem #4 Choose $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$ arbitrarily, then:

$$\begin{aligned} x \in \mathcal{P}(A) \cup \mathcal{P}(B) &\Rightarrow (x \in \mathcal{P}(A))\ or\ (x \in \mathcal{P}(B)) && (1)\\ &\Rightarrow (x \subseteq A)\ or\ (x \subseteq B) && (2)\\ &\Rightarrow x \subseteq (A \cup B) && (3)\\ &\Rightarrow x \in \mathcal{P}(A \cup B) && (4) \end{aligned}$$

Since our choice of x was arbitrary, we have that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Note that it is not true that these two are equal. To see this let $A = \{1\},\ B = \{2\},\ x = \{1,\ 2\}$. Clearly $x \in \mathcal{P}(A \cup B)$, but $x \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

This proof would not be accurate if we used $\Leftrightarrow$'s instead of $\Rightarrow$'s. In particular the proof would break down between steps (2) and (3) by the above example of A, B, and x.