Countable Borel Equivalence Relations
and
The Martin Conjecture

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Definition

A standard Borel space is a complete separable metric space equipped with its $\sigma$-algebra of Borel subsets.

Some Examples

- $\mathbb{R}$, $[0, 1]$, $2^\mathbb{N} = \mathcal{P}(\mathbb{N})$, ...
- If $\sigma$ is a sentence of $L_{\omega_1\omega}$, then

  $$\text{Mod}(\sigma) = \{ \mathcal{M} = \langle \mathbb{N}, \cdots \rangle \mid \mathcal{M} \models \sigma \}$$

  is a standard Borel space.
Let $G$ be the set of groups with underlying set $\mathbb{N}$.

We can identify each group $G \in G$ with the graph of its multiplication operation.

Then $G$ is a Borel subset of the Cantor space $2^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}$ and hence $G$ is a standard Borel space.

For later use, note that if $G, H \in G$, then

$$G \cong H \text{ iff } \exists \pi \in \text{Sym}(\mathbb{N}) \pi[m_G] = m_H.$$
Borel maps

**Definition**

Let $X$, $Y$ be standard Borel spaces.

- Then the map $\varphi : X \to Y$ is **Borel** iff $\text{graph}(\varphi)$ is a Borel subset of $X \times Y$.

- Equivalently, $\varphi : X \to Y$ is Borel iff $\varphi^{-1}(B)$ is a Borel set for each Borel set $B \subseteq Y$.

**An Analogue of Church’s Thesis**

\[ \text{EXPLICIT} = \text{BOREL} \]
Analytic and Borel equivalence relations

**Definition**

Let $E$ be an equivalence relation on the standard Borel space $X$.

- $E$ is **analytic** iff $E$ is an analytic subset of $X \times X$.
- $E$ is **Borel** iff $E$ is a Borel subset of $X \times X$.

**Example**

If $G, H \in \mathcal{G}$, then

$$G \cong H \text{ iff } \exists \pi \in \text{Sym}(\mathbb{N}) \quad \pi[m_G] = m_H.$$  

Hence $\cong_\mathcal{G}$ is an analytic equivalence relation.

**Theorem (Folklore)**

The isomorphism relation on $\mathcal{G}$ is analytic but **not** Borel.
The Polish space $\mathcal{G}_{fg}$ of f.g. groups

- A marked group $(G, \bar{s})$ consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators.

- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with $m$ distinguished generators.

- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

  $$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

- And $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ is the space of f.g. groups.
The Polish space $G_{fg}$ of f.g. groups

- Let $(G, \bar{s}) \in G_m$ and for each $\ell \geq 1$, let
  \[ B_\ell(G, \bar{s}) = \{ g \in G \mid \text{length}_S(g) \leq \ell \}. \]

- The basic open neighborhoods of $(G, \bar{s})$ in $G_m$ are given by
  \[ U_{(G,\bar{s}),\ell} = \{ (H, \bar{t}) \in G_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \}, \quad \ell \geq 1. \]

Example

For each $n \geq 1$, let $C_n = \langle g_n \rangle$ be cyclic of order $n$. Then:

\[ \lim_{n \to \infty} (C_n, g_n) = (\mathbb{Z}, 1). \]
A slight digression

Some Isolated Points
- Finite groups
- Finitely presented simple groups

The Next Stage
- $SL_3(\mathbb{Z})$

Question (Grigorchuk)
*What is the Cantor-Bendixson rank of $G_{fg}$?*
The isomorphism relation on $G_{fg}$ is a Borel equivalence relation.

In fact, the isomorphism relation on $G_{fg}$ is a countable Borel equivalence relation.

The Borel equivalence relation $E$ is countable iff every $E$-class is countable.
Countable Borel equivalence relations

Standard Example
Let $G$ be a countable (discrete) group and let $X$ be a standard Borel $G$-space. Then the corresponding orbit equivalence relation $E^X_G$ is a countable Borel equivalence relation.

Theorem (Feldman-Moore)
If $E$ is a countable Borel equivalence relation on the standard Borel space $X$, then there exists a countable group $G$ and a Borel action of $G$ on $X$ such that $E = E^X_G$. 
The Turing equivalence relation

**Definition**

The *Turing equivalence relation* $\equiv_T$ on $2^\mathbb{N}$ is defined by

$$A \equiv_T B \iff A \leq_T B \; \& \; B \leq_T A,$$

where $\leq_T$ denotes Turing reducibility.

**Remark**

Clearly $\equiv_T$ is a countable Borel equivalence relation on $2^\mathbb{N}$.

**Vague Question**

Can $\equiv_T$ be realised as the orbit equivalence relation of a “nice” Borel action of some countable group?
Borel reductions

**Definition**

Let $E$, $F$ be Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively.

- $E \leq_B F$ iff there exists a Borel map $f : X \to Y$ such that 
  \[ x \mathrel{E} y \iff f(x) \mathrel{F} f(y). \]

  In this case, $f$ is called a **Borel reduction** from $E$ to $F$.

- $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.

- $E <_B F$ iff both $E \leq_B F$ and $E \not\sim_B F$. 

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Some Examples

- Turing equivalence on $2^\mathbb{N}$.
- The isomorphism relations on the space $G_{fg}$ of f.g. groups.
- An interesting theory due to Dougherty, Harrington, Hjorth, Kechris, Jackson, Louveau,...
- Most basic questions remain open.
**Definition**

The Borel equivalence relation $E$ is smooth iff $E \leq_B \text{id}_{2^\mathbb{N}}$.

**Example**

The isomorphism relation for countable divisible abelian groups is smooth.

**Theorem**

The countable Borel equivalence relation $E$ on $X$ is smooth iff $X/E$ is a standard Borel space.
Countable Borel equivalence relations

\[ E_\infty = \text{universal} \]

\[ E_0 = \text{hyperfinite} \]

\[ id_{2^\mathbb{N}} = \text{smooth} \]

**Definition**

\( E_0 \) is the equivalence relation of eventual equality on the space \( 2^\mathbb{N} \) of infinite binary sequences.

**Theorem (DJK)**

If \( E \) is countable Borel, then \( E \) can be realized by a Borel \( \mathbb{Z} \)-action iff \( E \leq_B E_0 \).

**Theorem (Gao-Jackson)**

If \( G \) is a countable abelian group and \( X \) is a standard Borel \( G \)-space, then \( E_X^G \leq_B E_0 \).
Countable Borel equivalence relations

Definition

A countable Borel equivalence relation $E$ is **universal** iff $F \leq_B E$ for every countable Borel equivalence relation $F$.

Theorem (Thomas-Velickovic)

The isomorphism relation on $G_{fg}$ is countable universal.

Definition

A countable Borel equivalence relation $E_\infty$ is **universal** if $E_\infty$ is hyperfinite.

Definition

A countable Borel equivalence relation $E_0$ is **hyperfinite**.

Definition

A countable Borel equivalence relation $id_{2^\mathbb{N}}$ is **smooth**.
Countable Borel equivalence relations

Theorem (Adams-Kechris)
There exist $2^{\aleph_0}$ many countable Borel equivalence relations up to Borel bireducibility.

Question
Where does $\equiv_T$ fit into this picture?

Conjecture (Kechris)
$\equiv_T$ is universal.

Conjecture (Martin)
$\equiv_T$ is not universal.
The Turing degrees

**Definition**

The set of *Turing degrees* is defined to be

\[ \mathcal{D} = \{ a = [A]_\equiv_T \mid A \in 2^\mathbb{N} \}. \]

**Definition**

A subset \( X \subseteq \mathcal{D} \) is said to be **Borel** iff

\[ X^* = \bigcup \{ a \mid a \in X \} \]

is a Borel subset of \( 2^\mathbb{N} \).

**Remark**

\( \mathcal{D} \) is **not** a standard Borel space.
Example
For each \( a \in D \), the corresponding cone \( C_a = \{ b \in D \mid a \leq b \} \) is a Borel subset of \( D \).

Definition
If \( a, b \in D \), then \( a \leq b \) iff \( A \leq_T B \) for each \( A \in a \), \( B \in b \).

Theorem (Martin)
If \( X \subseteq D \) is Borel, then for some \( a \in D \), either \( C_a \subseteq X \) or \( C_a \subseteq D \setminus X \).
Suppose that $X \subseteq D$ is Borel and let $X^* = \bigcup \{ a \mid a \in X \} \subseteq 2^\mathbb{N}$.

Consider the two player Borel game $G(X^*)$

$$s(0) \ s(1) \ s(2) \ s(3) \ \cdots$$

where $I$ wins iff $s = (s(0) \ s(1) \ s(2) \ \cdots) \in X^*$.

Suppose that $\sigma : 2^{<\mathbb{N}} \rightarrow 2$ is a winning strategy for $I$.

Let $\sigma \leq_T t \in 2^\mathbb{N}$ and consider the run of $G(X^*)$ where

- $II$ plays $t = (s(1) \ s(3) \ s(5) \ \cdots)$
- $I$ responds with $\sigma$ and plays $(s(0) \ s(2) \ s(4) \ \cdots)$.

Then $s \in X^*$ and $s \equiv_T t$. Hence $C_\sigma \subseteq X$. 

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Martin’s Conjecture

**Definition**

A function \( \varphi : \mathcal{D} \rightarrow \mathcal{D} \) is Borel iff there exists a Borel function \( f : 2^\mathbb{N} \rightarrow 2^\mathbb{N} \) such that \( \varphi([A]_{\equiv_T}) = [f(A)]_{\equiv_T} \).

**Example**

The jump operator \( a \mapsto a' \) is a Borel function on \( \mathcal{D} \).

**Martin’s Conjecture**

If \( \varphi : \mathcal{D} \rightarrow \mathcal{D} \) is Borel, then either \( \varphi \) is constant on a cone or else \( \varphi(a) \geq a \) on a cone.
Remark
By Martin’s Theorem, if $\varphi : D \to D$ is Borel, then one of the following must occur on a cone:

1. $\varphi(a) < a$; or
2. $\varphi(a) \geq a$; or
3. $\varphi(a)$ and $a$ are incomparable.

Theorem (Slaman-Steel)

If $\varphi : D \to D$ is Borel and $\varphi(a) < a$ on a cone, then $\varphi$ is constant on a cone.
Some partial results

**Theorem (Slaman-Steel)**

If the Borel map \( \varphi : D \rightarrow D \) is uniformly invariant, then either \( \varphi \) is constant on a cone or else \( \varphi(a) \geq a \) on a cone.

**Slightly Inaccurate Definition**

A Borel function \( f : 2^\omega \rightarrow 2^\omega \) is uniformly invariant iff there exists a function \( t : \omega \times \omega \rightarrow \omega \times \omega \) such that “on a cone”

\[
A = \{ i \}^B, \quad B = \{ j \}^A \quad \implies \quad f(A) = \{ t_1(i, j) \}^{f(B)}, \quad f(B) = \{ t_2(i, j) \}^{f(A)}.
\]

**Conjecture (Steel)**

Every Borel map \( \varphi : D \rightarrow D \) is uniformly invariant.
Proposition (Martin’s Conjecture)

Suppose that $f : 2^\mathbb{N} \to 2^\mathbb{N}$ is a Borel reduction from $\equiv_T$ to $\equiv_T$ and let $\varphi : D \to D$ be the corresponding map. Then $\text{ran } \varphi$ contains a cone.

Proof.

Otherwise, there exists a cone $C_a$ such that

1. $C_a \cap \text{ran } \varphi = \emptyset$; and
2. $\varphi(b) \geq b$ for all $b \in C_a$.

But then $\varphi(a) \geq a$ and so $\varphi(a) \in C_a \cap \text{ran } \varphi$.

Corollary (Martin’s Conjecture)

1. $(\equiv_T \sqcup \equiv_T) \not\preccurlyeq_B \equiv_T$.
2. In particular, $\equiv_T$ is not countable universal.
Weak Borel Reducibility

Definition

Let $E, F$ be Borel equivalence relations on the standard Borel spaces $X, Y$ respectively.

- The map $f : X \to Y$ is a Borel homomorphism from $E$ to $F$ iff
  \[ x \ E \ y \iff f(x) \ F \ f(y). \]

- $E \leq^w_B F$ iff there exists a Borel homomorphism $f : X \to Y$ such that the induced map $X / E \to Y / F$ is countable-to-one.

- $E \sim^w_B F$ iff both $E \leq^w_B F$ and $F \leq^w_B E$.

- $E <^w_B F$ iff both $E \leq^w_B F$ and $E \sim^w_B F$. 

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Remark

- If $E$ and $F$ are countable Borel, then a Borel homomorphism $f : X \rightarrow Y$ is a weak Borel reduction iff $f$ is countable-to-one.
- In particular, if $E \subseteq F$ are countable Borel equivalence relations on $X$, then $E \leq^w_B F$ via the identity map.

Theorem (Kechris-Miller)

If $E, F$ are countable Borel equivalence relations on the uncountable standard Borel spaces $X, Y$ respectively, then the following are equivalent:

(a) $E \leq^w_B F$.
(b) There exists a countable Borel equivalence relation $S \subseteq F$ on $Y$ such that $E \sim_B S$. 

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Weakly universal equivalence relations

**Definition**
A countable Borel equivalence relation $E$ is said to be weakly universal iff $F \leq^w_B E$ for every countable Borel equivalence relation $F$.

**Proposition**
If $E$ is a countable Borel equivalence relation, then $E$ is weakly universal iff there exists a universal countable Borel equivalence relation $R \subseteq E$.

**Implausible Conjecture (Hjorth)**
Every weakly universal countable Borel equivalence relation is universal.
The weak universality of Turing equivalence

Theorem (Dougherty-Jackson-Kechris)

The orbit equivalence relation $E_\infty$ of translation action of the free group $\mathbb{F}_2$ on $2^{\mathbb{F}_2} = \mathcal{P}(\mathbb{F}_2)$ is countable universal.

Corollary

The Turing equivalence relation $\equiv_T$ is weakly universal.

Proof.

Identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$. 

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