361 PRACTICE SECOND MIDTERM QUESTIONS 2017

Question 1.

(a) Give the definition of $A \approx B$; i.e. the set A is equinumerous to the set B.

(b) Prove that if A is any set, then $A \not\approx \mathcal{P}(A)$.

Question 2.

(a) State the Cantor-Bernstein Theorem.

(b) A subset $S \subseteq \mathbb{N}$ is a *moiety* if both S and $\mathbb{N} \smallsetminus S$ are infinite. Let

$$\mathcal{M}(\mathbb{N}) = \{S \subseteq \mathbb{N} \mid S \text{ is a moiety } \}.$$

Prove that $\mathcal{M}(\mathbb{N}) \approx \mathcal{P}(\mathbb{N})$.

(c) Let $Fin(\mathbb{N})$ be the set of finite sets of natural numbers. Prove that $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{N}) \smallsetminus Fin(\mathbb{N})$.

Question 3.

(a) State the Cantor-Bernstein Theorem.

(b) Let $\mathrm{Fin}(\mathbb{N})$ be the set of finite subsets of $\mathbb{N}$. Prove that $\mathrm{Fin}(\mathbb{N}) \sim \mathbb{N}$.

(c) A function $f : \mathbb{N} \to \mathbb{N}$ is a *derangement* of $\mathbb{N}$ if f is a bijection such that $f(n) \neq n$ for all $n \in \mathbb{N}$. Determine whether the set $\mathrm{Der}(\mathbb{N})$ of derangements of $\mathbb{N}$ is countable or uncountable.

(d) A function $f : \mathbb{N} \to \mathbb{N}$ is said to be a *quasi-identity function* iff

$$|f(n) - n| \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Determine whether the set $QI(\mathbb{N})$ of quasi-identity functions is countable or uncountable.

Question 4.

(a) Define the addition operation $\kappa + \lambda$ for cardinal numbers κ and λ.

(b) Define the multiplication operation $\kappa \cdot \lambda$ for cardinal numbers κ and λ.

(c) Prove that $\aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$.

(d) Prove that if κ is a cardinal number such that $\kappa > 1$, then $\kappa + \kappa \leq \kappa \cdot \kappa$.

Question 5.

(a) Give the definition of a well-ordering $<$ of a set W.

(b) Let $\prec$ be the linear ordering on $\mathbb{N} \times \mathbb{Z}$ defined by $\langle a, b\rangle \prec \langle c, d\rangle$ if either:

- $a < c$ or
- $a = c$ and $b < d$.

Determine whether $\prec$ is a well-ordering of $\mathbb{N} \times \mathbb{Z}$.

Question 6. Let $\prec$ be the linear ordering on $\mathbb{N} \times \mathbb{N}$ defined by $\langle a, b\rangle \prec \langle c, d\rangle$ if either:

- $\max\{a, b\} < \max\{c, d\}$, or
- $\max\{a, b\} = \max\{c, d\}$ and $a < c$, or
- $\max\{a, b\} = \max\{c, d\}$ and $a = c$ and $b < d$.

Prove that $\prec$ is a well-ordering of $\mathbb{N} \times \mathbb{N}$. [*Note:* You do not need to prove that $\prec$ is a linear ordering of $\mathbb{N} \times \mathbb{N}$.]