The stationary set splitting game

Paul B. Larson

October 4, 2007
ω₁ is the least uncountable ordinal.

\( C \subset \omega_1 \) is club if it is closed and unbounded in the order topology on \( \omega_1 \).

\( A \subset \omega_1 \) is stationary if it intersects every club subset of \( \omega_1 \).
**Theorem 1** (Ulam). *Every stationary subset of \( \omega_1 \) can be split into \( \aleph_1 \) many stationary pieces.*
The stationary set splitting game $SG$ is a game of length $\omega_1$ between two players, Split and Unsplit.

In each round $\alpha$, Unsplit either accepts or rejects $\alpha$. If he accepts, then Split puts $\alpha$ into one of two sets ($A$ and $B$). Otherwise, Split does nothing.

After all $\omega_1$ rounds have been played, let $Y$ be the set of ordinals accepted by Unsplit. Split wins if $Y$ is nonstationary, or if both $A$ and $B$ are stationary.
• Martin’s Maximum implies that $SG$ is not determined.

• Split has a winning strategy in c.c.c. forcing extensions of models of “condensation.”

• It is possible to force a winning strategy for either player, and both are consistent with $\diamondsuit(S)$ holding for every stationary $S \subset \omega_1$. 
The canonical function game is a game of length $\omega_1$ between two players, Undominated and Dominating. In round $\alpha$, Undominated plays a countable ordinal $u(\alpha)$ and Dominating plays a wellordering $\sigma_\alpha$ of $\alpha$ of length greater than $u_\alpha$.

After all $\omega_1$ rounds have been played, Dominating wins if there is a club $C \subset \omega_1$ such that

$$\sigma_\alpha = \sigma_\beta \cap (\alpha \times \alpha)$$

for all $\alpha < \beta$ in $C$. 
The Neeman game for an \( n \)-ary formula \( \phi \) is a game of length \( \omega_1 \) where players \( I \) and \( II \) collaborate to build \( A \subset \omega_1 \), and \( I \) wins if there is a club \( C \subset \omega_1 \) such that
\[
\langle H(\omega_1), A, \in \rangle \models \phi(\alpha_1, \ldots, \alpha_n)
\]
for all \( \alpha_1 < \ldots < \alpha_n \) in \( C \).

Neeman has shown that the existence of an iterable model with indiscernible Woodin cardinals implies that all Neeman games for unary formulas are determined.

The payoff set for \( S_\mathcal{G} \) is a simple Boolean combination of payoffs for unary Neeman games.
Theorem 2 (Woodin). Suppose that there exists an iterable class model $M$ and a countable ordinal $\theta$ which is a Woodin limit of Woodin cardinals in $M$. Then there is a model in which all length-$\omega_1$ ordinal definable games on integers are determined.

3 Question. What are the consequences of this form of determinacy for combinatorics on $\omega_1$? ($\Diamond$? CH?)

4 Question. What about games on an arbitrary ordinal $\gamma$?
Theorem 5 (Woodin). If $\delta$ is a measurable Woodin cardinal, then in a forcing extension there exists an inner model satisfying all $\Sigma^2_2$ sentences $\phi$ such that $\phi + CH$ is forceable by a partial order in $V_\delta$.

The assertions that Unsplit or Split have a winning strategy in $SG$ are both $\Sigma^2_2(\text{NS}_{\omega_1})$, and both are consistent with $\diamond(S)$ holding for every stationary $S \subset \omega_1$. 
MLO is an extension of first-order logic with logical constants $=, \in$ and $\subseteq$ and a binary symbol $<$ as the only non-logical constant, allowing quantification over subsets of the domain. Every ordinal is a model for MLO, interpreting $<$ as $\in$.

Given an ordinal $\alpha$, an MLO game of length $\alpha$ is determined by an MLO formula with two free variables for subsets of the domain. In such a game, two players each build a subset of $\alpha$, and the winner is determined by whether these two sets satisfy the formula in $\alpha$. 
Theorem 6 (Büchi-Landweber). All MLO games of length $\omega$ are determined.

Theorem 7 (Rabinovich). All MLO games of countable length are determined.

The stationary set splitting game is an example of an MLO game of length $\omega_1$ whose determinacy is independent of ZFC.
Indeterminacy from MM

Given a strategy for either player, consider the two-step forcing adding a generic run against the strategy by initial segments, and then forcing a club witnessing that the player using the strategy lost.

Either this two-step forcing preserves stationary sets from the ground model, or the strategy can already be defeated in the ground model.
A collection $S$ of subsets of a nonempty stationary set $X$ is stationary if for all

$$F: X^{<\omega} \to X$$

there is a member of $S$ closed under $F$.

When $\omega_1 \subset X$, $S$ is projective stationary if for all stationary $E \subset \omega_1$, the set

$$\{a \in S \mid a \cap \omega_1 \in E\}$$

is stationary.
\( C^+ \) is the statement that there exists a projective stationary set \( \mathcal{X} \) consisting of countable elementary substructures of \( H(\aleph_2) \) such that for all \( X, Y \) in \( \mathcal{X} \) with

\[
X \cap \omega_1 = Y \cap \omega_1,
\]
either every for every club \( C \subset \omega_1 \) in \( X \) there is a club \( D \subset \omega_1 \) in \( Y \) with

\[
D \cap X \subset C \cap X,
\]
or for every for every club \( D \subset \omega_1 \) in \( Y \) there is a club \( C \subset \omega_1 \) in \( X \) with

\[
C \cap X \subset D \cap X.
\]
$\mathcal{C}+$ implies that Split has a winning strategy:

Let

$$y, a, b \in \mathcal{P}(\alpha)$$

be the result of the first $\alpha$ rounds of the game, and suppose that Unsplit has accepted $\alpha$. Suppose that there exists an $X \in \mathcal{X}$ with

$$X \cap \omega_1 = \alpha$$

and

$$Y, A, B \in \mathcal{P}(\omega_1) \cap X$$

such that $Y$ is stationary and the disjoint union of $A$ and $B$ and

$$y = Y \cap \alpha,$$

$$a = A \cap \alpha$$ and $$b = B \cap \alpha.$$ Then if there is a club $C \in X \cap \mathcal{P}(\omega_1)$ such that

$$C \cap A = \emptyset,$$

put $\alpha \in A$. If there is a club $C \in X \cap \mathcal{P}(\omega_1)$ such that $C \cap B = \emptyset$, put $\alpha \in B$. 

15
\( \mathcal{C}^+ \) holds in models of “condensation” (including \( L \)) and is preserved by c.c.c. forcing extensions.

Justin Moore has shown that it is incompatible with PFA.
$\mathcal{D}_u$ is the statement that there exists a $\diamondsuit$-sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ such that for all $A \subset \omega_1$ there is a club $C \subset \omega_1$ such that either

- for all $\alpha \in C$, $a_\alpha = A \cap \alpha \rightarrow \alpha \in C$ or

- for all $\alpha \in C$, $a_\alpha = A \cap \alpha \rightarrow \alpha \not\in C$
$\mathcal{D}_u$ implies that Unsplit has a winning strategy: put $\alpha$ in $Y$ if and only if $a_\alpha = A \cap \alpha$.

To force $\mathcal{D}_u$, first add a sequence

$$\langle a_\alpha \subset \alpha : \alpha < \omega_1 \rangle$$

by initial segments. Then iteratively destroy the stationarity of any stationary subset of $\omega_1$ witnessing that this sequence does not witness $\mathcal{D}_u$. 
strong club guessing:

A sequence $\langle c_\alpha : \alpha < \omega_1 \text{ limit} \rangle$ is a strong club guessing sequence if each $c_\alpha$ is a cofinal subset of $\alpha$ and for all club $C \subset \omega_1$, the set of $\alpha$ such that $c_\alpha \setminus C$ is finite contains a club.
(too) strong club guessing:

\[ \langle c_\alpha : \alpha < \omega_1 \lim \rangle \text{ such that each } c_\alpha \text{ is a cofinal subset of } \alpha \text{ and for all club } C \subset \omega_1 \text{ and all stationary } A \subset \omega_1, \text{ there exists an } \alpha \in A \text{ such that } c_\alpha \setminus C \text{ is finite and } c_\alpha \cap A \text{ is infinite.} \]

Define \( A \) recursively by: \( \alpha \in A \) if and only if \( A \cap \alpha \cap c_\alpha \) is finite.
\( C_s \) is the statement that there exist
\[
\langle a^\alpha_\beta : \alpha < \omega_1 \text{ limit, } \beta < \gamma_\alpha \rangle
\]
such that

- each \( \gamma_\alpha \) is a countable ordinal,
- each \( a^\alpha_{\beta} \) is a cofinal subset of \( \alpha \),
- \( \beta < \beta' < \gamma_\alpha \) implies that \( a^\alpha_{\beta'} \setminus a^\alpha_{\beta} \) is finite,
- for all club \( C \subset \omega_1 \) and all stationary \( A \subset \omega_1 \), there exist \( \alpha \in A \) and \( \beta < \gamma_\alpha \) such that \( a^\alpha_{\beta} \setminus C \) is finite and \( a^\alpha_{\beta} \cap A \) is infinite.
\( C_s \) implies that Split has a winning strategy.

To force \( C_s \), first add a sequence

\[
\langle a^\alpha_\beta : \alpha < \omega_1 \text{ limit}, \beta < \gamma_\alpha \rangle
\]

by initial segments. Then iteratively destroy the stationarity of any stationary subset of \( \omega_1 \) witnessing that this sequence does not witness \( C_s \).