Question 1.  
(a) Give the definition of $\Gamma \models \varphi$ and $\Gamma \vdash \varphi$.

(b) State the Soundness Theorem.

(c) Let $L$ have the following nonlogical symbols:

(i) a binary predicate symbol $<; and

(iii) a unary function symbol $f$.

Let $T$ be the theory in $L$ with the following axioms:

1. $(\forall x) \neg (x < x)$
2. $(\forall x)(\forall y)(x < y \lor y < x \lor x = y)$
3. $(\forall x)(\forall y)(\forall z)(x < y \land y < z \rightarrow [x < z])$
4. $(\forall x)(\forall y)(\forall z)(x < y \rightarrow (\exists z)[x < z \land z < y])$
5. $(\forall x)(\exists y)(\exists z)(y < x \land x < z)$
6. $(\forall x)(\forall y)(x < y \rightarrow [fx < fy])$. 
7. $(\forall x)(x < fx)$.

Let $\sigma$ be the sentence

$$(\forall x)(\exists y)(fy = x).$$

Prove that $T \not\vdash \sigma$ and that $T \not\vdash \neg \sigma$.

Question 2.  
(a) State the Soundness Theorem and the Completeness Theorem for predicate logic.

(b) State and prove the Los-Vaught Theorem.

(c) Let $L$ have the following nonlogical symbols:

- a binary predicate symbol $E$; and
- two unary predicate symbols $P$ and $Q$.

For each $n \geq 1$, let $\varphi_n$ be the sentence which says that there exist at least $n$ distinct elements satisfying the predicate $P$. For example, $\varphi_2$ is the sentence:

$$(\exists x)(\exists y)(x \neq y \land Px \land Py).$$
Similarly, for each \( n \geq 1 \), let \( \psi_n \) be the sentence which says that there exist at least \( n \) distinct elements satisfying the predicate \( Q \). Let \( T \) be the theory in \( \mathcal{L} \) with the following axioms:

- \( (\forall x)(Px \lor Qx) \)
- \( (\forall x)\neg(Px \land Qx) \)
- \( \varphi_n \) for all \( n \geq 1 \).
- \( \psi_n \) for all \( n \geq 1 \).
- \( (\forall x)(\forall y)(Exy \iff (Px \land Qy)) \)

Prove that \( T \) is consistent and complete.