Popa Superrigidity and Countable Borel Equivalence Relations II

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A quick recap

The Fundamental Question

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. To what extent does the data $(X, E^X_G)$ “remember” the group $G$ and its action on $X$?

More accurately, to what extent does the data $(C, E^X_G | C)$ “remember” the group $G$ and its action on $X$, where $C$ is an arbitrary Borel complete section?

Further Hypotheses

We shall usually also assume that:

- $G$ acts freely on $X$; i.e. $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$.
- There exists a $G$-invariant probability measure $\mu$ on $X$. 

Ergodicity

**Definition**

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. Then the $G$-invariant probability measure $\mu$ is said to be **ergodic** iff $\mu(A) = 0, 1$ for every $G$-invariant Borel subset $A \subseteq X$.

**Theorem**

If $\mu$ is a $G$-invariant probability measure on the standard Borel $G$-space $X$, then the following statements are equivalent.

- The action of $G$ on $(X, \mu)$ is ergodic.
- If $Y$ is a standard Borel space and $f : X \to Y$ is a $G$-invariant Borel function, then there exists a $G$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function.
**Strong mixing**

**Definition**

The action of $G$ on the standard probability space $(X, \mu)$ is **strongly mixing** iff for any Borel subsets $A, B \subseteq X$, we have that

$$\mu(g(A) \cap B) \to \mu(A) \cdot \mu(B) \quad \text{as } g \to \infty.$$ 

In other words, if $\langle g_n \mid n \in \mathbb{N} \rangle$ is a sequence of distinct elements of $G$, then

$$\lim_{n \to \infty} \mu(g_n(A) \cap B) = \mu(A) \cdot \mu(B).$$

**Observation**

If $H \leq G$ is an infinite subgroup of $G$, then the action of $H$ on $(X, \mu)$ is also strongly mixing.
Strong mixing continued

Observation

If the action of $G$ on $(X, \mu)$ is strongly mixing, then $G$ acts ergodically on $(X, \mu)$.

Proof.

If $A \subseteq X$ is a $G$-invariant Borel subset, then

$$\mu(A)^2 = \lim_{g \to \infty} \mu(g(A) \cap A) = \lim_{g \to \infty} \mu(A) = \mu(A).$$

Hence $\mu(A) = 0, 1$.

Remark

With more effort, it can be shown that for each $n \geq 2$, the diagonal action of $G$ on $(X^n, \mu^n)$ is also ergodic.
Bernoulli actions are strongly mixing

**Theorem**

The action of $G$ on $((2)^G, \mu)$ is strongly mixing.

- Consider the case when there exist finite subsets $S, T \subset G$ and subsets $\mathcal{F} \subseteq 2^S$, $\mathcal{G} \subseteq 2^T$ such that $A = \{ f \in (2)^G \mid f \upharpoonright S \in \mathcal{F} \}$ and $B = \{ f \in (2)^G \mid f \upharpoonright T \in \mathcal{G} \}$.

- If $\langle g_n \mid n \in \mathbb{N} \rangle$ is a sequence of distinct elements of $G$, then

  $$g_n(S) \cap T = \emptyset$$

  for all but finitely many $n$.

- This means that $g_n(A)$, $B$ are independent events and so

  $$\mu(g_n(A) \cap B) = \mu(g_n(A)) \cdot \mu(B) = \mu(A) \cdot \mu(B).$$
Let $G$ be a countable group and let $X$ be a standard Borel $G$-space with invariant ergodic probability measure $\mu$. Suppose that the countable group $H$ has a free Borel action on $Y$ and that

$$f : X \rightarrow Y$$

is a Borel homomorphism between the corresponding orbit equivalence relations. Then we can define a Borel cocycle

$$\alpha : G \times X \rightarrow H$$

by setting

$$\alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$
The cocycle identity

Note that

\[ f(x) \xrightarrow{\alpha(g,x)} f(g \cdot x) \xrightarrow{\alpha(h,g \cdot x)} f(hg \cdot x) \]

and hence we have the identity:

\[ \alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \quad \mu\text{-a.e } x \]

In particular, \( f \) is a permutation group homomorphism iff

\[ \alpha(g, x) = \alpha(g) \]

is a group homomorphism.
Cocycle equivalence

\[
\beta(g, x) = b(g \cdot x)\alpha(g, x)b(x)^{-1} \quad \mu\text{-a.e } x
\]
Theorem (Popa)

Let $\Gamma$ be a countably infinite Kazhdan group and $G, \mathbb{G}$ be countable groups such that $\Gamma \leq G \leq \mathbb{G}$. If $H$ is any countable group, then every Borel cocycle

$$\alpha : G \times (2)^\mathbb{G} \rightarrow H$$

is equivalent to a group homomorphism of $G$ into $H$.

Remarks

- In applications, we usually have $G = \mathbb{G}$.
- For example, we let $\Gamma = \text{SL}_3(\mathbb{Z})$ or any subgroup of finite index in $\text{SL}_3(\mathbb{Z})$.
- For example, we can let $G = \mathbb{G} = \Gamma \times S$, where $S$ is any countable group.
An easy consequence of Popa Superrigidity

Definition

\[ E_G \text{ denotes the orbit equivalence relation of the Bernoulli action of the countable group on } ((2)^G, \mu). \]

Theorem

- Let \( G = SL_3(\mathbb{Z}) \times S \), where \( S \) is any countable group.
- Let \( H \) be any countable group and let \( Y \) be a free standard Borel \( H \)-space.

If there exists a \( \mu \)-nontrivial Borel homomorphism from \( E_G \) to \( E_Y^H \), then there exists a virtual embedding \( \pi : G \rightarrow H \).
Proof of Theorem

Suppose that \( f : (2)^G \rightarrow Y \) is a \( \mu \)-nontrivial Borel homomorphism from \( E_G \) to \( E_H^Y \).

Then we can define a Borel cocycle \( \alpha : G \times (2)^G \rightarrow H \) by

\[
\alpha(g, x) = \text{the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).
\]

By Popa, after deleting a nullset and slightly adjusting \( f \), we can suppose that \( \alpha : G \rightarrow H \) is a group homomorphism.

Suppose that \( N = \ker \alpha \) is infinite.

Since the action of \( G \) is strongly mixing, it follows that \( N \) acts ergodically on \( ((2)^G, \mu) \).

But then the \( N \)-invariant function \( f : (2)^G \rightarrow X \) is \( \mu \)-a.e. constant, which is a contradiction.
A map of the world

Essentially Free

Turing Equivalence

$E_\infty$

$E_0$
Some open problems

Open Question (Thomas)
Suppose that $E$ is a countable Borel equivalence relation on the standard Borel space $X$ with invariant ergodic probability measure $\mu$. Does there always exist a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is essentially free?

Open Question (Hjorth)
Suppose that $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space $X$. If $E$ is universal, does it follow that $F$ is necessarily also universal?

Theorem (Adams)
There exist countable Borel equivalence relations $E \subseteq F$ such that $E, F$ are incomparable with respect to Borel reducibility.
Unique ergodicity

Definition

The action of $G$ on the standard probability space $(X, \mu)$ is uniquely ergodic iff $\mu$ is the unique $G$-invariant probability measure on $X$.

Observation

If the action of $G$ on $(X, \mu)$ is uniquely ergodic, then $G$ acts ergodically on $(X, \mu)$.

Proof.

- Suppose that there exists a $G$-invariant Borel subset $A \subseteq X$ with $0 < \mu(A) < 1$. Let $B = X \setminus A$.
- Then we can define distinct $G$-invariant probability measures by
  
  \[ \nu_1(Z) = \frac{\mu(Z \cap A)}{\mu(A)} \]
  \[ \nu_2(Z) = \frac{\mu(Z \cap B)}{\mu(B)} \]
Examples of unique ergodicity

Theorem (Farrell-Varadarajan)

If $G$ is a countable group and $(X, \mu)$ is a standard Borel $G$-space with invariant ergodic probability measure $\mu$, then there exists a $G$-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that the action on $G$ on $(X_0, \mu)$ is uniquely ergodic.

Remark

In other words, every ergodic action is almost uniquely ergodic.
Corollary (Hjorth-Kechris)

Suppose that $G$ is a countable group and that $(X, \mu)$ is a standard Borel $G$-space with invariant probability measure $\mu$. If the action of $G$ on $(X, \mu)$ is strongly mixing, then there exists a $G$-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that the action of every infinite f.g. subgroup of $G$ on $(X_0, \mu)$ is uniquely ergodic.

Proof.

- If $L$ is an infinite f.g. subgroup of $G$, then $L$ acts ergodically on $(X, \mu)$.
- Hence there exists an $L$-invariant Borel subset $Y_L \subseteq X$ with $\mu(Y_L) = 0$ such that the action on $L$ on $(X \setminus Y_L, \mu)$ is uniquely ergodic.
- Let $X_L = X \setminus G \cdot Y_L$. Then $X_0 = \bigcap_L X_L$ satisfies our requirements.
Adams’ theorem

Theorem (Adams)

There exist countable Borel equivalence relations $E \subseteq F$ such that $E, F$ are incomparable with respect to Borel reducibility.

- From now on, let $S = SL_3(\mathbb{Z})$ and let $T$ be a proper subgroup of finite index. For example, we could let $T$ be the kernel of the homomorphism $\varphi : SL_3(\mathbb{Z}) \to SL_3(\mathbb{F}_7)$.
- Then $T$ is also a Kazhdan group and so we can apply Popa’s Cocycle Superrigidity Theorem to the actions of both $S$ and $T$ on $((2)^S, \mu)$.
- Let $X \subseteq (2)^S$ be an $S$-invariant Borel subset with $\mu(X) = 1$ such that the action of every infinite f.g. subgroup of $S$ on $(X, \mu)$ is uniquely ergodic.
- Let $E \subseteq F$ be the orbit equivalence relations corresponding to the free actions of $T \leq S$ on $(X, \mu)$. 
To see that $F \not\leq_B E$

Applying Popa superrigidity, if $F \leq_B E$, then there exists a virtual embedding $\pi : S = SL_3(\mathbb{Z}) \to T$.

Hence the result follows from:

Lemma

Suppose that $G$ is a (not necessarily proper) subgroup of finite index in $SL_3(\mathbb{Z})$. Then:

- $G$ has no nontrivial finite normal subgroups.
- $G$ doesn’t embed into any of its proper subgroups of finite index.
To see that $E \not\preceq_B F$

- Suppose that $f : X \to X$ is a Borel reduction from $E$ to $F$.
- Then we can define a corresponding Borel cocycle 
  $\alpha : T \times X \to S$ by

  $$\alpha(t, x) = \text{the unique } s \in S \text{ such that } s \cdot f(x) = f(t \cdot x).$$

- By Popa, after deleting a nullset and slightly adjusting $f$, we can suppose that $\alpha : T \to S$ is a group homomorphism.
- Since $T$ has no finite normal subgroups, it follows that $\alpha$ is an embedding. Since $S \not\cong T$, it follows that $\alpha(T)$ is a proper subgroup of $S$.
- Since the actions of $S$, $T$ on $(X, \mu)$ are free and
  $$\alpha(t) \cdot f(x) = f(t \cdot x) \quad \text{for } t \in T, x \in X,$$

  it follows that $f$ is also an injection.
Adams’ trick

Thus we have an embedding \((T, X) \xrightarrow{\alpha, f} (S, X)\) of permutation groups and so we can define an \(\alpha(T)\)-invariant probability measure \(\nu = f_* \mu\) on \(X\) by \(\nu(A) = \mu(f^{-1}(A))\).

By unique ergodicity, we must have that \(\nu = \mu\) and hence \(\mu(f(X)) = 1\).

Let \(s \in S \setminus \alpha(T)\). We claim that \(f(X) \cap s \cdot f(X) = \emptyset\), which is a contradiction.

Suppose that \(f(x) = s \cdot f(y) \in f(X) \cap s \cdot f(X)\). Then \(f(x) \not\sim f(y)\) and so there exists \(t \in T\) such that \(x = t \cdot y\). Hence

\[
\alpha(t) \cdot f(y) = f(t \cdot y) = f(x) = s \cdot f(y)
\]

and so \(s^{-1} \alpha(t) \cdot f(y) = f(y)\), which contradicts the fact that \(S\) acts freely on \(X\).
A final application

Definition
An additive subgroup $G \leq \mathbb{Q}^n$ has rank $n$ iff $G$ contains $n$ linearly independent elements.

Theorem
For each $n \geq 1$, the isomorphism relation on the space of torsion-free abelian groups of rank $n$ is not countable universal.

A slightly embarrassing question
Is the isomorphism relation on the space of torsion-free abelian groups of finite rank countable universal?

Answer
Of course not! But why? By Popa Superrigidity.