Popa Superrigidity and Countable Borel Equivalence Relations I

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Standard Borel Spaces

Definition

A standard Borel space is a Polish space $X$ equipped with its $\sigma$-algebra of Borel subsets.

Some Examples

- $\mathbb{R}$, $[0, 1]$, $2^\mathbb{N}$, $\mathbb{Q}_p$, ...
- If $\sigma$ is a sentence of $L_{\omega_1\omega}$, then
  \[ \text{Mod}(\sigma) = \{ \mathcal{M} = \langle \mathbb{N}, \cdots \rangle \mid \mathcal{M} \models \sigma \} \]
  is a standard Borel space.

Theorem (Kuratowski)

There exists a unique uncountable standard Borel space up to isomorphism.
Borel maps

Definition

Let $X$, $Y$ be standard Borel spaces.

- Then the map $\varphi : X \to Y$ is Borel iff $\text{graph}(\varphi)$ is a Borel subset of $X \times Y$.
- Equivalently, $\varphi : X \to Y$ is Borel iff $\varphi^{-1}(B)$ is a Borel set for each Borel set $B \subseteq Y$.

Church’s Thesis for Real Mathematics

$\text{EXPLICIT} = \text{BOREL}$
Borel equivalence relations

**Definition**

Let $X$ be a standard Borel space. Then a **Borel equivalence relation** on $X$ is an equivalence relation $E \subseteq X^2$ which is a Borel subset of $X^2$.

**Definition**

Let $G$ be a Polish group. Then a **standard Borel $G$-space** is a standard Borel space $X$ equipped with a Borel action $(g, x) \mapsto g \cdot x$. The corresponding $G$-orbit equivalence relation is denoted by $E_X^G$.

**Observation**

If $G$ is a countable (discrete) group and $X$ is a standard Borel $G$-space, then $E_X^G$ is a Borel equivalence relation.
Definition

Let $E$, $F$ be Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively.

- $E \leq_B F$ iff there exists a Borel map $f : X \to Y$ such that

  $$x E y \iff f(x) F f(y).$$

In this case, $f$ is called a \textbf{Borel reduction} from $E$ to $F$.

- $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.

- $E <_B F$ iff both $E \leq_B F$ and $E \not\sim_B F$.

Definition

More generally, $f : X \to Y$ is a \textbf{Borel homomorphism} from $E$ to $F$ iff

$$x E y \implies f(x) F f(y).$$
Milestones

\[ E_{K_\sigma} = \text{universal } K_\sigma \]

\[ E_1 = \text{hypersmooth} \]

\[ E_0 = \text{hyperfinite} \]

\[ E_\infty = \text{universal countable} \]

\[ \text{id}_\mathbb{R} = \text{smooth} \]

Countable Borel equivalence relations
Countable Borel equivalence relations

**Definition**

Let $E$ be a Borel equivalence relation.

- $E$ is **countable** iff every $E$-class is countable.
- $E$ is **essentially countable** iff there exists a countable Borel equivalence relation $F$ such that $E \sim_B F$.

**Standard Example**

Let $G$ be a countable (discrete) group and let $X$ be a standard Borel $G$-space. Then the corresponding orbit equivalence relation $E^X_G$ is a countable Borel equivalence relation.

**Theorem (Feldman-Moore)**

If $E$ is a countable Borel equivalence relation on the standard Borel space $X$, then there exists a countable group $G$ and a Borel action of $G$ on $X$ such that $E = E^X_G$. 
Some Examples

- Turing equivalence on $\mathcal{P}(\mathbb{N})$.
- The isomorphism relations for classes of countable structures which are "finitely generated" in some broad sense.
- An interesting theory due to Dougherty, Harrington, Hjorth, Kechris, Jackson, Louveau, ...
- Most basic questions remain open.
Countable Borel equivalence relations

Definition
The Borel equivalence relation $E$ is smooth iff $E \leq_B \text{id}_{2^\mathbb{N}}$, where $2^\mathbb{N}$ is the space of infinite binary sequences.

Example
The isomorphism relation on the space of connected locally finite graphs with transitive automorphism groups.

$E_\infty = \text{universal}$

$E_0$

$id_{2^\mathbb{N}} = \text{smooth}$
Definition

$E_0$ is the equivalence relation of eventual equality on the space $2^\mathbb{N}$ of infinite binary sequences.

Theorem (DJK)

If $E$ is countable Borel, then $E$ can be realized by a Borel $\mathbb{Z}$-action iff $E \leq_B E_0$.

Theorem (Jackson-Gao)

If $G$ is a countable abelian group and $X$ is a standard Borel $G$-space, then $E^X_G \leq_B E_0$. 

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Countable Borel equivalence relations

Definition
A countable Borel equivalence relation $E$ is *universal* iff $F \leq_B E$ for every countable Borel equivalence relation $F$.

Theorem (JKL)
The orbit equivalence relation $E_\infty$ of the action of the free group $\mathbb{F}_2$ on its powerset $\mathcal{P}(\mathbb{F}_2)$ is countable universal.
Countable Borel equivalence relations

Theorem (Adams-Kechris 2000)

There exist $2^\aleph_0$ many countable Borel equivalence relations up to Borel bireducibility.

$E_0 = \text{universal}$

Uncountably many relations

$id_{2^\mathbb{N}} = \text{smooth}$
Comparing orbit equivalence relations

Stating the obvious

If $G, H$ are countable groups and $X, Y$ are a standard Borel $G$-space, $H$-space respectively, then the following are equivalent:

- $E^X_G \leq B E^Y_H$.
- There exist a Borel map $f : X \rightarrow Y$ such that for all $a, b \in X$,
  $$G \cdot a = G \cdot b \iff H \cdot f(a) = H \cdot f(b).$$

The Fundamental Question

- Does the complexity of $E^X_G$ reflect the structural complexity of the group $G$?
- To what extent does the data $(X, E^X_G)$ “remember” $G$ and its action on $X$?
An easy counterexample ...

- For each countable group $G$, consider the Borel action of $G$ on $G \times [0, 1]$ defined by $g \cdot (h, r) = (gh, r)$.
- Then the Borel map $(h, r) \mapsto (1_G, r)$ selects a point in each $G$-orbit, and so the corresponding orbit equivalence relation is smooth.

Observation

If $G$ acts freely on $X$ and preserves a probability measure, then $E^X_G$ isn’t smooth.

Definition

The Borel action of the countable group $G$ on the standard Borel space $X$ is free iff $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$. 

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Theorem (Dougherty-Jackson-Kechris)

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. If $X$ does not admit a $G$-invariant probability measure, then for every countable group $H \supseteq G$, there exists a Borel action of $H$ on $X$ such that $E^X_H = E^X_G$.

Theorem

If $E$ is a countable aperiodic Borel equivalence relation, then $E$ can be realised as the orbit equivalence relation of a faithful Borel action of uncountably many countable groups.

Definition

A countable Borel equivalence relation $E$ is aperiodic iff every $E$-class is infinite.
Question

Let \( E \) be a nonsmooth countable Borel equivalence relation. Does there necessarily exist a countable group \( G \) with a free measure-preserving Borel action on a standard probability space \((X, \mu)\) such that \( E \sim_B E^X_G \)?

Theorem (Dougherty-Jackson-Kechris)

Suppose that \( E \) is a countable Borel equivalence relation on an uncountable standard Borel space. Then there exists a countable group \( G \) and a standard Borel \( G \)-space \( X \) such that:

- \( G \) preserves a nonatomic probability measure \( \mu \) on \( X \).
- \( E \sim_B E^X_G \).
**Free actions**

**Definition**

- The Borel action of the countable group $G$ on the standard Borel space $X$ is **free** iff $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$. In this case, we say that $X$ is a **free standard Borel $G$-space**.

- The countable Borel equivalence relation $E$ on $X$ is **free** iff there exists a countable group $G$ with a free Borel action on $X$ such that $E^X_G = E$.

- The countable Borel equivalence relation $E$ is **essentially free** iff there exists a free countable Borel equivalence relation $F$ such that $E \sim_B F$.

**Question (Jackson-Kechris-Louveau)**

*Is every countable Borel equivalence relation essentially free?*
Some closure properties

**Theorem (Jackson-Kechris-Louveau)**

Let $E$, $F$ be countable Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively.

- If $E \leq_B F$ and $F$ is essentially free, then so is $E$.
- If $E \subseteq F$ and $F$ is essentially free, then so is $E$.

**Theorem**

Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$. Then $E$ is essentially free if $E$ can be realised by a Borel action of:

- (Jackson-Kechris-Louveau) a f.g. group of polynomial growth.
- (Gao-Jackson) an arbitrary countable abelian group.
- (Thomas) a Tarski monster.
Bernoulli actions

- Let $G$ be a countably infinite group and consider the shift action on $\mathcal{P}(G) = 2^G$.
- Then the usual product probability measure $\mu$ on $2^G$ is $G$-invariant and the free part of the action

$$\mathcal{P}^*(G) = (2)^G = \{ x \in 2^G \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G \}$$

has $\mu$-measure 1.
- Let $E_G$ be the corresponding orbit equivalence relation on $(2)^G$.

Observation

*If* $G \leq H$, *then* $E_G \leq_B E_H$.

Proof.

The inclusion map $\mathcal{P}^*(G) \hookrightarrow \mathcal{P}^*(H)$ is a Borel reduction from $E_G$ to $E_H$. 

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Homomorphisms

Definition

Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ with invariant probability measure $\mu$.

Let $F$ be a countable Borel equivalence relation on the standard Borel space $Y$.

Then the Borel homomorphism $f : X \rightarrow Y$ from $E$ to $F$ is said to be $\mu$-trivial iff there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that $f$ maps $Z$ into a single $F$-class.

Definition

If $G$, $H$ are countable groups, then the group homomorphism $\pi : G \rightarrow H$ is a virtual embedding iff $|\ker \pi| < \infty$. 
An easy consequence of Popa superrigidity

**Theorem**

- Let $G = SL_3(\mathbb{Z}) \times S$, where $S$ is any countable group.
- Let $H$ be any countable group and let $Y$ be a free standard Borel $H$-space.

If there exists a $\mu$-nontrivial Borel homomorphism from $E_G$ to $E_Y^H$, then there exists a virtual embedding $\pi : G \to H$.

**Remark**

In particular, the conclusion holds if there exists a Borel subset $Z \subseteq (2)^G$ with $\mu(Z) = 1$ such that $E_G \upharpoonright Z \leq_B E_Y^H$. 

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Theorem

If $E$ is an essentially free countable Borel equivalence relation, then there exists a countable group $G$ such that $E_G \not\preceq_B E$.

Corollary

The class of essentially free countable Borel equivalence relations does not admit a universal element. In particular, $E_\infty$ is not essentially free.
We can suppose that $E = E^X_H$ is realised by a free Borel action on $X$ of the countable group $H$.

Let $L$ be a finitely generated group which does not embed into $H$.

Let $S = L \ast \mathbb{Z}$ and let $G = SL_3(\mathbb{Z}) \times S$.

Then $G$ has no finite normal subgroups and so there does not exist a virtual embedding $\pi : G \to H$.

Hence $E_G \not\preceq_B E^X_H$. 

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A map of the world

Essentially Free

Turing Equivalence

$E_0$

$E_\infty$
Uncountably many free countable Borel equivalence relations

**Definition**

- For each prime $p \in \mathbb{P}$, let $A_p = \bigoplus_{i=0}^{\infty} C_p$.
- For each subset $C \subseteq \mathbb{P}$, let
  $$G_C = SL_3(\mathbb{Z}) \times \bigoplus_{p \in C} A_p.$$ 

**Theorem**

If $C, D \subseteq \mathbb{P}$, then $E_{G_C} \leq_B E_{G_D}$ iff $C \subseteq D$. 

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Ergodicity

**Definition**

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. Then the $G$-invariant probability measure $\mu$ is said to be **ergodic** iff $\mu(A) = 0, 1$ for every $G$-invariant Borel subset $A \subseteq X$.

**Example**

Every countable group $G$ acts ergodically on $((2)^G, \mu)$.

**Theorem**

If $\mu$ is a $G$-invariant probability measure on the standard Borel $G$-space $X$, then the following statements are equivalent.

- The action of $G$ on $(X, \mu)$ is ergodic.
- If $Y$ is a standard Borel space and $f : X \rightarrow Y$ is a $G$-invariant Borel function, then there exists a $G$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function.
Towards uncountably many non-essentially free countable Borel equivalence relations

Definition

The countable groups $G$, $H$ are **virtually isomorphic** iff there exist finite normal subgroups $N \trianglelefteq G$, $M \trianglelefteq H$ such that $G/N \cong H/M$.

Lemma

There exists a Borel family $\{S_x \mid x \in 2^\mathbb{N}\}$ of f.g. groups such that if $G_x = SL_3(\mathbb{Z}) \times S_x$, then the following conditions hold:

- If $x \neq y$, then $G_x$ and $G_y$ are not virtually isomorphic.
- If $x \neq y$, then $G_x$ doesn’t virtually embed in $G_y$.

Definition

For each Borel subset $A \subseteq 2^\mathbb{N}$, let $E_A = \bigcup_{x \in A} E_{G_x}$. 

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Lemma

If the Borel subset $A \subseteq 2^\mathbb{N}$ is uncountable, then $E_A$ is not essentially free.

Proof.

- Suppose that $E_A \leq_B E_H^Y$, where $H$ is a countable group and $Y$ is a free standard Borel $H$-space.
- Then for each $x \in A$, we have that $E_{G_x} \leq_B E_H^Y$ and so there exists a virtual embedding $\pi_x : G_x \to H$.
- Since $A$ is uncountable, there exist $x \neq y \in A$ such that $\pi_x[G_x] = \pi_y[G_y]$.
- But then $G_x, G_y$ are virtually isomorphic, which is a contradiction.
Uncountably many non-essentially free relations

Lemma

\[ E_A \leq_B E_B \iff A \subseteq B. \]

Proof.

- Suppose that \( E_A \leq_B E_B \).
- Suppose also that \( A \nsubseteq B \) and that \( x \in A \setminus B \).
- Then there exists a Borel reduction from \( E_{G_x} \) to \( E_A \)

\[
f : (2)^{G_x} \to \bigsqcup_{y \in B} (2)^{G_y}.
\]

- By ergodicity, there exists \( \mu_x \)-measure 1 subset of \( (2)^{G_x} \) which maps to a fixed \( (2)^{G_y} \).
- This yields a \( \mu_x \)-nontrivial Borel homomorphism from \( E_{G_x} \) to \( E_{G_y} \) and so \( G_x \) virtually embeds into \( G_y \), which is a contradiction.