Open Colourings
and automorphisms of the Calkin algebra

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$\mathcal{B}(H)$: The algebra of all bounded operators on a complex Hilbert space $H$.

Adjoint, $a^*$:

$$(a^* \xi | \eta) = (\xi | a \eta)$$

$(\mathcal{B}(H), +, \cdot, *, \| \cdot \|)$ is a Banach algebra with involution.
The Calkin algebra

\[ K(H) = \{ a \mid \{ a\xi \mid \|\xi\| \leq 1 \} \text{ is compact} \} = \{ a \mid \{ a\xi \mid \|\xi\| \leq 1 \} \text{ is compact} \} \]

\[ C(H) = B(H)/K(H) \] is the Calkin algebra.

\[ \pi : B(H) \rightarrow C(H) : \text{The quotient map.} \]
Fact

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An automorphism $\Phi$ of $\mathcal{P}(\mathbb{N})/\text{Fin}$ is *trivial* if it is represented by $f : \mathbb{N} \to \mathbb{N}$, so that

$$[X]_{\text{Fin}} \mapsto [f^{-1}[X]]_{\text{Fin}}.$$
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**Theorem (W. Rudin, 1957)**

$\mathsf{CH}$ implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^{2^{\aleph_0}}$ (mostly nontrivial) automorphisms.
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$CH$ implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^{2^\aleph_0}$ (mostly nontrivial) automorphisms.

**Theorem (S. Shelah, 1979)**

It is relatively consistent with ZFC that all automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ are trivial.
If $\Psi_1$ and $\Psi_2$ are automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$, then so is $\Psi_1 \oplus \Psi_2$
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An automorphism of $\mathcal{C}(H)$ is inner iff its restriction to $\mathcal{C}(H_0)$ for some (any) infinite-dimensional subspace $H_0$ of $H$ is inner.
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In $\mathcal{C}(H)$ there are $2^{\aleph_0}$ many orbits.
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**Theorem (Junge–Pisier, 1995)**

1. $\delta_{cb}$ is separable on subspaces of any separable algebra.
2. $\delta_{cb}$ is nonseparable on three-dimensional subspaces of $\mathcal{C}(H)$.
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**Corollary (Phillips, 2000)**

*No separable subalgebra of $\mathcal{C}(H)$ realizes all 3-types.*
Theorem (Phillips–Weaver, 2006)

CH implies there is an outer automorphism of the Calkin algebra.
Todorcevic’s OCA:
If $G = (V, E)$ is a graph such that $E = \bigcup_{n=0}^{\infty} U_n \times V_n$, then $G$ is either countably chromatic or it has an uncountable clique.
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Theorem (Farah, 2007)

*Todorcevic’s OCA implies all automorphisms of $\mathcal{C}(H)$ are inner.*
\((E_n)_{n \in \mathbb{N}}\): an orthogonal decomposition of \(H\) into finite-dimensional subspaces.

\[
\mathcal{D}[\tilde{E}] = \{ a \in \mathcal{B}(H) \mid a[E_n] \subseteq E_n \text{ for all } n \}
\]

\[
\mathcal{C}[\tilde{E}] = \mathcal{D}[\tilde{E}] / \mathcal{K}[\tilde{E}].
\]
Fix an automorphism $\Phi$ of $\mathcal{C}(H)$.

**Lemma**

*OCA implies that $\Phi$ is inner on each $\mathcal{C}[\vec{E}]$.***
Fix an automorphism $\Phi$ of $C(H)$.

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*OCA implies that if $\Phi$ is inner on each $C[\vec{E}]$ then it is inner.*
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Proposition (Farah, Geschke)

$Assume b = \aleph_1$ and $2^{\aleph_1} > 2^{\aleph_0}$. Then there is an outer automorphism of $\mathcal{C}(H)$ that is inner on each $\mathcal{C}[\vec{E}]$. 
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*PFA implies that all automorphisms of $C(H)$ for any complex Hilbert space $H$ are inner.*
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Theorem (Veličković)

PFA implies all automorphisms of $P(\kappa)/\text{Fin}$ are trivial.
Open problems

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Question

*Is it consistent with ZFC that there is an automorphism $\Phi$ of $\mathcal{C}(H)$ and $a \in \mathcal{C}(H)$ such that $\Phi(a) \neq uau^*$ for all $u$?
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Theorem (Brown–Douglas–Fillmore)

Not if $a = \pi(b)$ for a normal $b$. ($b$ is normal if $bb^* = b^*b$.)
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*PFA implies that all automorphisms of* $C(H)$ *for any complex Hilbert space* $H$ *are inner.*

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*Is it consistent with ZFC that there is an automorphism* $\Phi$ *of* $C(H)$ *and* $a \in C(H)$ *such that* $\Phi(a) \neq uau^*$ *for all* $u$?

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*Not known for a normal* $a$. 