A Locally Finite Maximal Cofinitary Group

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Outline

- Definitions and Basics.
- Motivation.
- The Result and Idea.
Definition
Sym(\(\mathbb{N}\)): the group of bijections \(\mathbb{N} \to \mathbb{N}\) with operation composition.

\(f \in \text{Sym}(\mathbb{N})\) is cofinitary iff \(f\) is the identity or has only finitely many fixed points.

\(G \leq \text{Sym}(\mathbb{N})\) is a cofinitary group (sharp group) iff all \(g \in G\) are cofinitary.

\(G \leq \text{Sym}(\mathbb{N})\) is a maximal cofinitary group (MCG) iff \(G\) is a cofinitary group and is not properly contained in another cofinitary group.
(Adeleke, Truss) A maximal cofinitary group can not be countable.

(P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.

Any cofinitary group is contained in a maximal cofinitary group.

(Yi Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG $G$ with $|G| = \kappa$. 

A cofinitary group is an almost disjoint (eventually different) family of permutations that is also a group.
Theorem (Su Gao, Yi Zhang)

\( V = L \) implies there is an \( mcg \) with a coanalytic generating set.

Theorem

\( V = L \) implies there is a coanalytic \( mcg \).
Question
Does there exist an analytic mcg?
Theorem (Otmar Spinas)
There does not exist a locally compact mcg.

Theorem
There does not exist an eventually bounded mcg.
Constructing MCG

Use CH or MA.

\[ G_{\alpha+1} = \langle G_{\alpha}, g_{\alpha} \rangle = (G_{\alpha} * F(x))[x := g_{\alpha}] \]

\[ g_{\alpha} = \bigcup_{s \in \mathbb{N}} g_{\alpha,s} \]

Study

\[ w(g_{\alpha,s}) \leadsto w(g_{\alpha,s+1}) \]

*good extension*: no unavoidable new fixed points.

\[ w = u^{-1}vu \]

“Always” gives *free groups*. 
Theorem

*MA implies there exists an mcg into which every countable group embeds.*

\[ G = \ast_{\alpha < c} G_{\alpha}. \]

with \( G_{\alpha} \) of all different isomorphism types.
Question
Are the possible complexities of free mcg the same as the possible complexities of all mcg?
Theorem

MA implies there exists a locally finite mcg.
A finite cg has action on \( \mathbb{N} \) with all finite orbits. On all but finitely many of these orbits it acts regularly (transitive and fixed point free).

Any regular action of a group is isomorphic to the group acting on itself.
Have $G = \{g_n : n < \omega\}$ a locally finite cofinitary group. Let $G_n$ denote $\langle g_i : i \leq n \rangle$.

$f : \mathbb{N} \rightarrow \mathbb{N}$ a finite injective map. Extend $f$ and make act nice with larger and larger parts of $G$. 
Lemma
Let $G$ be a finite cofinitary group, and $h \in \text{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, h \rangle$ is cofinitary. Then for all finite subgroups $H \leq G$ and all but finitely many $n \in \mathbb{N}$ the numbers $n$ and $h(n)$ are not in the same $H$ orbit.