Question 1. Let $F_2$ be the free group on two generators. Prove that if $P \subseteq F_2$ is an arbitrary subset, then there exists a finite subset $Q \subseteq P$ such that for all homomorphisms $\varphi, \psi : F_2 \to F_2$, if $\varphi \upharpoonright Q = \psi \upharpoonright Q$, then $\varphi \upharpoonright P = \psi \upharpoonright P$.

Some hints:
(1) Regard $F_2$ as the subgroup of $SL_2(\mathbb{Z})$ freely generated by
$$a = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$ 
(2) Let $R$ be the polynomial ring $\mathbb{Z}[x_1, \ldots, x_8, y_1, \ldots, y_8]$ and define
$$M_a = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad M_b = \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix}$$
and
$$N_a = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \quad N_b = \begin{pmatrix} y_5 & y_6 \\ y_7 & y_8 \end{pmatrix}.$$ 
Then for each $w = w(a, b) \in F_2$, there are corresponding matrices $M_w, N_w$ obtained as suitable products of the matrices $M_a, M_b, N_a, N_b$.

(3) Note that if $\varphi, \psi : F_2 \to F_2$ are homomorphisms, then $\varphi(a), \varphi(b)$ can be obtained by substituting appropriate values $\varphi(x_i) \in \mathbb{Z}$ into $M_a, M_b$. Then if $w = w(a, b) \in F_2$, we see that $\varphi(w)$ is the result of substituting $\varphi(x_i)$ into $M_w$. Similarly, $\psi(a), \psi(b)$ can be obtained by substituting appropriate values $\psi(y_i) \in \mathbb{Z}$ into $N_a, N_b$, etc.

(4) Given a subset $P \subset F_2$, consider the ideal of $R$ corresponding to the set of matrix equations
$$\{M_w = N_w \mid w \in P\}.$$

Definition 1. If $\varphi \in \text{Aut}(F_2)$, then the fixed subgroup is
$$\text{Fix}(\varphi) = \{w \in F_2 \mid \varphi(w) = w\}.$$
and the recurrent subgroup is

\[ \text{Rec}(\varphi) = \{ w \in F_2 \mid \varphi^k(w) = w \text{ for some } k \geq 1 \}. \]

**Question 2.** Prove that if \( \varphi \in \text{Aut}(F_2) \), then there exists \( k \geq 1 \) such that \( \text{Rec}(\varphi) = \text{Fix}(\varphi^k) \).

**Hint:** Apply Question 1 to the subset

\[ \text{Rec}(\varphi) = \bigcup_{n \geq 1} \{ w \in F_2 \mid \varphi^n(w) = id(w) \}. \]