Question 1. Compute the Galois group over $\mathbb{Q}$ of the polynomial

$$x^6 + 22x^5 - 9x^4 + 12X^3 - 37x^2 - 29x - 15$$

(Hint: Reduce mod 2,3,5.)

Question 2. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$ be an irreducible polynomial with roots $\pm \alpha, \pm \beta$ and splitting field $K$.

(a) Prove that the Galois group $G = \text{Aut}_{\mathbb{Q}} K$ is isomorphic to a transitive subgroup of the Sylow 2-subgroup $D_4$ of $\text{Sym}(4)$; and hence $G$ is isomorphic to one of the following groups:

(i) $\mathbb{Z}/4\mathbb{Z}$ (ii) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (iii) $D_4$.

(b) Prove that:

- Case (i) holds if and only if $\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \in \mathbb{Q}$.
- Case (ii) holds if and only if $\alpha \beta \in \mathbb{Q}$.
- Otherwise, case (iii) holds.

Question 3. Suppose that $K$ is a field of characteristic 0 and that $\sigma \in \text{Aut}_K K^{\text{alg}}$. Let

$$F = \{ u \in K^{\text{alg}} \mid \sigma(u) = u \}$$

be the corresponding fixed field. Prove that every finite extension of $F$ is cyclic.