The quasi-isometry relation for finitely generated groups

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F.g. groups viewed as metric spaces

Some motivation (Gromov)

- Introducing geometric language to bring amplification and clarification.
- Suggesting new concepts and constructions.
- Extending the class of applicable ideas changes the class of essential examples.
F.g. groups viewed as metric spaces

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- Introducing geometric language to bring amplification and clarification.
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Confession (Gromov)

“After all, the actual reason why one approaches a problem from a geometric angle is because one’s mind is bent this way. No amount of rationalization can conceal the truth.”
Cayley graphs of finitely generated groups

**Definition**

Let $G$ be a finitely generated group and let $S \subseteq G \setminus \{1_G\}$ be a finite generating set. Then the **Cayley graph** $\text{Cay}(G, S)$ is the graph with vertex set $G$ and edge set

$$E = \{\{x, y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$ 

The corresponding **word metric** is denoted by $d_S$.

For example, when $G = \mathbb{Z}$ and $S = \{1\}$, then the corresponding Cayley graph is:

```
-2 -1  0  1  2
```

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But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then the corresponding Cayley graph is:

![Cayley graph diagram]

Theorem (S.T.)

There does not exist a Borel choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

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But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then the corresponding Cayley graph is:

![Cayley graph diagram]

**Theorem (S.T.)**

There does not exist a *Borel* choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.
The basic idea of geometric group theory

Although the Cayley graphs of a f.g. group $G$ with respect to different generating sets $S$ are usually nonisomorphic, they always have the same large scale geometry.
The quasi-isometry relation

**Definition (Gromov)**

Let $G, H$ be f.g. groups with word metrics $d_S, d_T$ respectively. Then $G, H$ are said to be quasi-isometric, written $G \approx_Q H$, iff there exist

- constants $\lambda \geq 1$ and $C \geq 0$, and
- a map $\varphi : G \to H$

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$
Definition (Gromov)

Let $G, H$ be f.g. groups with word metrics $d_S, d_T$ respectively. Then $G, H$ are said to be **Lipschitz equivalent** iff there exist

- a constant $\lambda \geq 1$, and
- a map $\varphi : G \rightarrow H$

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y);$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) = 0.$$
The quasi-isometry relation

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Let $G, H$ be f.g. groups with word metrics $d_S, d_T$ respectively. Then $G, H$ are said to be quasi-isometric, written $G \approx_Q H$, iff there exist

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and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

Of course, this notion makes sense for arbitrary metric spaces, including connected graphs with their path metrics.
As expected ...

Observation

If $S, S'$ are finite generating sets for $G$, then

$$id : \langle G, d_S \rangle \rightarrow \langle G, d_{S'} \rangle$$

is a quasi-isometry.

Thus while it doesn’t make sense to talk about the isomorphism type of the Cayley graph of $G$, it does make sense to talk about the quasi-isometry type.
Theorem (Gromov)

If $G, H$ are f.g. groups, then the following are equivalent.

- $G$ and $H$ are quasi-isometric.
- There exists a locally compact space $X$ on which $G, H$ have commuting proper actions via homeomorphisms such that $X/G$ and $X/H$ are both compact.
A topological criterion

Theorem (Gromov)

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Definition

The action of the discrete group $G$ on $X$ is proper iff for every compact subset $K \subseteq X$, the set \( \{ g \in G \mid g(K) \cap K \neq \emptyset \} \) is finite.
**Obviously quasi-isometric groups**

**Definition**

Two groups $G_1$, $G_2$ are said to be **virtually isomorphic**, written $G_1 \approx_v G_2$, iff there exist subgroups $N_i \leq H_i \leq G_i$ such that:

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- $N_1, N_2$ are finite normal subgroups of $H_1, H_2$ respectively.
- $H_1/N_1 \cong H_2/N_2$.

**Proposition (Folklore)**

*If the f.g. groups $G_1, G_2$ are virtually isomorphic, then $G_1, G_2$ are quasi-isometric.*
Theorem (Folklore)

Let $\Gamma, \Delta$ be irreducible and reducible cocompact lattices in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Then $\Gamma$ and $\Delta$ are quasi-isometric but not virtually isomorphic.

Theorem (Dioubina)

The f.g. groups $\text{Alt}(5) \wr \mathbb{Z}$ and $C_{60} \wr \mathbb{Z}$ are quasi-isometric but not virtually isomorphic. (In fact, they have isomorphic Cayley graphs.)
Theorem (Grigorchuk-Bowditch)

There are $2^\aleph_0$ f.g. groups up to quasi-isometry.
Growth rates and quasi-isometric groups

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Proof (Grigorchuk).

Consider the growth rate of the size of balls of radius $n$ in the Cayley graphs of suitably chosen groups.
Theorem (Grigorchuk-Bowditch)

There are $2^\aleph_0$ f.g. groups up to quasi-isometry.

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Proof (Bowditch).

Consider the growth rate of the length of “irreducible loops” in the Cayley graphs of suitably chosen groups.
Question

What are the possible complete invariants for the quasi-isometry problem for f.g. groups?
The complexity of the quasi-isometry relation

Question

What are the possible complete invariants for the quasi-isometry problem for f.g. groups?

Question

Is the quasi-isometry problem for f.g. groups strictly harder than the isomorphism problem?
Let $E$, $F$ be Borel equivalence relations on the Polish spaces $X$, $Y$.

- $E \leq_B F$ iff there exists a Borel map $f : X \to Y$ such that
  
  $$x E y \iff f(x) F f(y).$$

- $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.

- $E <_B F$ iff both $E \leq_B F$ and $E \not\sim_B F$. 

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Let $F_m$ be the free group on $\{x_1, \cdots, x_m\}$ and let $G_m$ be the compact space of normal subgroups of $F_m$. Since each $m$-generator group can be realised as a quotient $F_m/N$ for some $N \in G_m$, we can regard $G_m$ as the space of $m$-generator groups. There are natural embeddings

\[
G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_m \hookrightarrow \cdots
\]

and we can regard

\[
G = \bigcup_{m \geq 1} G_m
\]

as the space of f.g. groups.
A slight digression

Theorem (Champetier-Guirardel)

If $G$ is a finitely generated group, then the following are equivalent:

- $G$ is a limit of free groups in some compact space $\mathcal{G}_m$.
- $G$ has the same universal theory as a free group.

Question (Grigorchuk)

What is the Cantor-Bendixson rank of $G_m$?

Question (Ghys)

Does there exist a nonatomic $\sim$-invariant ergodic probability measure on $G_m$?
A slight digression

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The isomorphism relation

Theorem (Champetier)

The isomorphism relation $\simeq$ on the space $\mathcal{G}$ of f.g. groups is a countable Borel equivalence relation.
The isomorphism relation

**Theorem (Champetier)**

The isomorphism relation \( \cong \) on the space \( \mathcal{G} \) of f.g. groups is a countable Borel equivalence relation.

The natural action of the countable group \( \text{Aut}(\mathbb{F}_m) \) on \( \mathbb{F}_m \) induces a corresponding homeomorphic action on the compact space \( \mathcal{G}_m \) of normal subgroups of \( \mathbb{F}_m \). Furthermore, each \( \pi \in \text{Aut}(\mathbb{F}_m) \) extends to a homeomorphism of the space \( \mathcal{G} \) of f.g. groups.

Clearly if \( N, M \in \mathcal{G}_m \) and there exists \( \pi \in \text{Aut}(\mathbb{F}_m) \) such that \( \pi(N) = M \), then \( \mathbb{F}_m/N \cong \mathbb{F}_m/M \). Unfortunately, the converse does not hold.
The isomorphism relation continued

Theorem (Tietze)

If $N, M \in \mathcal{G}_m$, then the following are equivalent:

- $F_m/N \cong F_m/M$.
- There exists $\pi \in \text{Aut}(F_{2m})$ such that $\pi(N) = M$. 

Corollary (Champetier)
The isomorphism relation $\cong$ on the space $\mathcal{G}_m$ of f.g. groups is the orbit equivalence relation arising from the homeomorphic action of the countable group $\text{Aut}_{\text{f}}(F_{\infty})$ of finitary automorphisms of the free group $F_{\infty}$ on $\{x_1, x_2, \ldots, x_m, \ldots\}$. 

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The isomorphism relation continued

Theorem (Tietze)

If $N, M \in \mathcal{G}_m$, then the following are equivalent:

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A universal countable Borel equivalence relation

Confirming a conjecture of Hjorth-Kechris ...

**Theorem (S.T.-Velickovic)**

The isomorphism relation $\simeq$ on the space $\mathcal{G}$ of f.g. groups is a universal countable Borel equivalence relation.
A universal countable Borel equivalence relation

Confirming a conjecture of Hjorth-Kechris ...

**Theorem (S.T.-Velickovic)**

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**Remark**

The proof shows that the isomorphism relation on the space $\mathcal{G}_5$ of 5-generator groups is already countable universal. Presumably the same is true for the isomorphism relation on $\mathcal{G}_2$?
The equivalence relation $E$ on the Polish space $X$ is $K_\sigma$ iff $E$ is the union of countably many compact subsets of $X \times X$. 
**Definition**

The equivalence relation $E$ on the Polish space $X$ is $K_\sigma$ iff $E$ is the union of countably many compact subsets of $X \times X$.

**Example**

The following are $K_\sigma$ equivalence relations on the space $G$ of f.g. groups:

- the isomorphism relation $\cong$
- the virtual isomorphism relation $\approx_v$
- the quasi-isometry relation $\approx_Q$
A $\mathcal{K}_\sigma$ equivalence relation is not necessarily the union of an increasing countable chain of compact equivalence relations.
Warning

A $K_\sigma$ equivalence relation is not necessarily the union of an increasing countable chain of compact equivalence relations.

Observation

If $E$ is the union of an increasing countable chain of compact equivalence relations, then $E$ is hypersmooth.

Definition

A Borel equivalence relation $E$ is smooth iff $E$ is Borel reducible to $id_\mathbb{R}$. 
The Kechris-Louveau classification

**Theorem (Kechris-Louveau)**

*Up to Borel bireducibility, $E_0$ and $E_1$ are the only nonsmooth hypersmooth Borel equivalence relations.*
The Kechris-Louveau classification

**Theorem (Kechris-Louveau)**

Up to Borel bireducibility, $E_0$ and $E_1$ are the only nonsmooth hypersmooth Borel equivalence relations.

**Definition**

$E_0$ is the Borel equivalence relation on $2^\mathbb{N}$ defined by

$$x E_0 y \iff x(n) = y(n) \text{ for almost all } n.$$

**Definition**

$E_1$ is the Borel equivalence relation on $(2^\mathbb{N})^\mathbb{N}$ defined by

$$x E_1 y \iff x(n) = y(n) \text{ for almost all } n.$$
Some universal $K_\sigma$ equivalence relations

Theorem (Rosendal)

Let $E_{K_\sigma}$ be the equivalence relation on $\prod_{n \geq 1}[1, n]$ defined by

$$\alpha E_{K_\sigma} \beta \iff \exists N \forall k |\alpha(k) - \beta(k)| \leq N.$$ 

Then $E_{K_\sigma}$ is a universal $K_\sigma$ equivalence relation.
Theorem (Rosendal)

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Theorem (Rosendal)

The Lipschitz equivalence relation on the standard Borel space of compact separable metric spaces is Borel bireducible with $E_{K_\sigma}$. 
More universal $K_\sigma$ equivalence relations

**Theorem (S.T.)**

The following equivalence relations are Borel bireducible with $E_{K_\sigma}$

- the growth rate relation on the space of strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$;
- the quasi-isometry relation on the space of connected graphs of bounded degree.

**Definition**

The strictly increasing functions $f, g : \mathbb{N} \to \mathbb{N}$ have the same growth rate, written $f \equiv g$, iff there exists an integer $t \geq 1$ such that

- $f(n) \leq g(tn)$ for all $n \geq 1$, and
- $g(n) \leq f(tn)$ for all $n \geq 1$. 
The quasi-isometry problem

Conjecture

The quasi-isometry problem for f.g. groups is Borel bireducible with $E_{\mathcal{K}_\sigma}$. 

Theorem (S.T.)

The virtual isomorphism problem for f.g. groups is strictly harder than the isomorphism problem.

Theorem (S.T.)

The quasi-isometry problem for f.g. groups isn't smooth.
Conjecture

The quasi-isometry problem for f.g. groups is Borel bireducible with $E_{K_{\sigma}}$.

Theorem (S.T.)

The virtual isomorphism problem for f.g. groups is strictly harder than the isomorphism problem.
Conjecture

The quasi-isometry problem for f.g. groups is Borel bireducible with $E_{K_0}$.

Theorem (S.T.)

The virtual isomorphism problem for f.g. groups is strictly harder than the isomorphism problem.

Theorem (S.T.)

The quasi-isometry problem for f.g. groups isn’t smooth.
The main conjectures

\[ E_{K_{\sigma}} \cong \text{quasi-isometry for f.g. groups} \]

virtual isomorphism for f.g. groups

\[ E_{1} \]

\[ E_{0} \]

\[ E_{\infty} = \text{isomorphism for f.g. groups} \]

\[ id_{\mathbb{R}} \]
The virtual isomorphism relation

Definition

The f.g. groups $G_1$, $G_2$ are virtually isomorphic, written $G_1 \simeq \hat{V} G_2$, iff there exist subgroups $N_i \leq H_i \leq G_i$ such that:

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- $N_1, N_2$ are finite normal subgroups of $H_1, H_2$ respectively.
- $H_1/N_1 \cong H_2/N_2$. 

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The virtual isomorphism relation

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- $N_1$, $N_2$ are finite normal subgroups of $H_1$, $H_2$ respectively.
- $H_1/N_1 \cong H_2/N_2$.

**Definition**

The f.g. groups $G_1$, $G_2$ are (abstractly) **commensurable**, written $G_1 \approx_C G_2$, iff there exist subgroups $H_i \leq G_i$ of finite index such that $H_1 \cong H_2$. 

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The virtual isomorphism relation

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**Definition**

The f.g. groups $G_1$, $G_2$ are **quasi-isomorphic**, written $G_1 \approx_F G_2$, iff there exist finite normal subgroups $N_i \leq G_i$ such that $G_1/N_1 \cong G_2/N_1$. 

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The commensurability relation

**Theorem (S.T.)**

The commensurability relation $\approx_C$ on the space $\mathcal{G}$ of f.g. groups is a universal countable Borel equivalence relation.

Proof. For each f.g. group $G$, there are only countably many groups up to isomorphism which are commensurable with $G$. Fix some infinite f.g. simple group $S$. Then if $G, H$ are any groups, we have that $G \approx H$ iff $(\text{Alt}(5) \wr G) \wr S \approx C (\text{Alt}(5) \wr H) \wr S$.
The commensurability relation

Theorem (S.T.)

The commensurability relation $\sim_C$ on the space $\mathcal{G}$ of f.g. groups is a universal countable Borel equivalence relation.

Proof.

For each f.g. group $G$, there are only countably many groups up to isomorphism which are commensurable with $G$. 
The commensurability relation

**Theorem (S.T.)**

The commensurability relation $\approx_C$ on the space $\mathcal{G}$ of f.g. groups is a universal countable Borel equivalence relation.

**Proof.**

- For each f.g. group $G$, there are only countably many groups up to isomorphism which are commensurable with $G$.
- Fix some infinite f.g. simple group $S$. Then if $G, H$ are any groups, we have that

  $$G \cong H \iff (\text{Alt}(5) \wr G) \wr S \approx_C (\text{Alt}(5) \wr H) \wr S.$$
Find the reduction

Corollary

There exists a Borel reduction from $\approx_C$ to $\equiv$. 
Find the reduction

Corollary

There exists a Borel reduction from $\approx_C$ to $\cong$.

Open Problem

Find a "group-theoretic" reduction from $\approx_C$ to $\cong$. 
Corollary

There exists a Borel reduction from $\approx_C$ to $\cong$.

Open Problem

Find a “group-theoretic” reduction from $\approx_C$ to $\cong$.

Theorem (S.T.)

There does not exist a Borel reduction $f$ from $\approx_C$ to $\cong$ such that $f(G) \approx_C G$ for all $G \in \mathcal{G}$.

In other words, there is no Borel way of selecting an isomorphism class within each commensurability class.
There exists a Borel map \( s \mapsto G_s \) from \( 2^\mathbb{N} \) to \( \mathcal{G} \) such that:

- Each \( G_s \) is a Grigorchuk group.
- \( G_s \cong G_t \) if and only if \( s = t \).
- If \( s \in E_0 t \), then \( G_s \approx C G_t \).
- If \( s \notin E_0 t \), then \( G_s \) and \( G_t \) have different growth rates.

Thus if \( s, t \in 2^\mathbb{N} \), then \( s \in E_0 t \) if and only if \( G_s \approx C G_t \) if and only if \( G_s \approx Q G_t \).
There exists a Borel map $s \mapsto G_s$ from $2^\mathbb{N}$ to $\mathcal{G}$ such that:

- Each $G_s$ is a Grigorchuk group.
Sketch Proof of Theorem

There exists a Borel map \( s \mapsto G_s \) from \( 2^\mathbb{N} \) to \( \mathcal{G} \) such that:

- Each \( G_s \) is a Grigorchuk group.
- \( G_s \cong G_t \) iff \( s = t \).

Corollary

The quasi-isometry relation \( \cong_Q \) is not smooth.
Sketch Proof of Theorem

There exists a Borel map $s \mapsto G_s$ from $2^\mathbb{N}$ to $\mathcal{G}$ such that:

- Each $G_s$ is a Grigorchuk group.
- $G_s \cong G_t$ iff $s = t$.
- If $s \in E_0 t$, then $G_s \approx_C G_t$.

Corollary

The quasi-isometry relation $\approx_Q$ is not smooth.
Sketch Proof of Theorem

There exists a Borel map $s \mapsto G_s$ from $2^\mathbb{N}$ to $\mathcal{G}$ such that:

- Each $G_s$ is a Grigorchuk group.
- $G_s \cong G_t$ iff $s = t$.
- If $s E_0 t$, then $G_s \approx_C G_t$.
- If $s \not\sim E_0 t$, then $G_s, G_t$ have different growth rates.
There exists a Borel map $s \mapsto G_s$ from $2^\mathbb{N}$ to $\mathcal{G}$ such that:

- Each $G_s$ is a Grigorchuk group.
- $G_s \cong G_t$ iff $s = t$.
- If $s \not\sim_0 t$, then $G_s \cong_C G_t$.
- If $s \not\sim_0 t$, then $G_s, G_t$ have different growth rates.

Thus if $s, t \in 2^\mathbb{N}$, then

$$s \sim_0 t \iff G_s \cong_C G_t \iff G_s \cong_Q G_t.$$
Sketch Proof of Theorem

There exists a Borel map \( s \mapsto G_s \) from \( 2^\mathbb{N} \) to \( \mathcal{G} \) such that:

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- If \( s \not\in E_0 t \), then \( G_s \cong_C G_t \).
- If \( s \not\in E_0 t \), then \( G_s, G_t \) have different growth rates.

Thus if \( s, t \in 2^\mathbb{N} \), then

\[
  s \in E_0 t \quad \text{iff} \quad G_s \cong_C G_t \quad \text{iff} \quad G_s \cong_Q G_t.
\]

Corollary

The quasi-isometry relation \( \cong_Q \) is not smooth.
The quasi-isomorphism relation

**Definition**

The f.g groups $G_1$, $G_2$ are quasi-isomorphic, written $G_1 \approx_F G_2$, iff there exist finite normal subgroups $N_i \trianglelefteq G_i$ such that $G_1/N_1 \cong G_2/N_1$.

**Theorem (S.T.)**

$E_1 \times E_\infty \leq B \approx F \leq B E K \sigma$.

**Conjecture**

Both of the above inequalities are strict.

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The f.g groups $G_1$, $G_2$ are **quasi-isomorphic**, written $G_1 \approx_F G_2$, iff there exist finite normal subgroups $N_i \triangleleft G_i$ such that $G_1/N_1 \cong G_2/N_1$.

**Theorem (S.T.)**

$E_1 \times E_\infty \leq_B \approx_F \leq_B E_{K_\sigma}$. 
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**Theorem (S.T.)**

$E_1 \times E_\infty \leq_B \approx_F \leq_B E_{K_\sigma}.$

**Conjecture**

*Both of the above inequalities are strict.*
How many $K_{\sigma}$ relations?

Theorem

\[ E_1 \sqcup E_\infty <_B E_1 \times E_\infty \leq_B E_{K_{\sigma}}. \]
How many $K_\sigma$ relations?

**Theorem**

\[ E_1 \sqcup E_\infty \prec_B E_1 \times E_\infty \leq_B E_{K_\sigma}. \]

**Proof.**

- Let $\mu, \nu$ be the usual product probability measures on $2^{F_2}, (2^N)^N$. 

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How many $K_\sigma$ relations?

**Theorem**

$$E_1 \sqcup E_\infty <_B E_1 \times E_\infty \leq_B E_{K_\sigma}.$$ 

**Proof.**

- Let $\mu, \nu$ be the usual product probability measures on $2^{F_2}, (2^N)^N$.
- Suppose that $f : (2^N)^N \times 2^{F_2} \to (2^N)^N \sqcup 2^{F_2}$ is a Borel reduction from $E_1 \times E_\infty$ to $E_1 \sqcup E_\infty$. 

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How many $K_\sigma$ relations?

**Theorem**

$$E_1 \sqcup E_\infty <_B E_1 \times E_\infty \leq_B E_{K_\sigma}.$$ 

**Proof.**

- Let $\mu$, $\nu$ be the usual product probability measures on $2^{2^\mathbb{N}}$, $(2^\mathbb{N})^\mathbb{N}$.
- Suppose that $f : (2^\mathbb{N})^\mathbb{N} \times 2^{2^\mathbb{N}} \to (2^\mathbb{N})^\mathbb{N} \sqcup 2^{2^\mathbb{N}}$ is a Borel reduction from $E_1 \times E_\infty$ to $E_1 \sqcup E_\infty$.
- Then for all $x \in (2^\mathbb{N})^\mathbb{N}$, $f(x, y) \in 2^{2^\mathbb{N}}$ for $\mu$-a.e. $y \in 2^{2^\mathbb{N}}$. 

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How many $K_\sigma$ relations?

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\[ E_1 \sqcup E_\infty <_B E_1 \times E_\infty \leq_B E_{K_\sigma}. \]

**Proof.**

- Let $\mu, \nu$ be the usual product probability measures on $2^{F_2}, (2^N)^N$.
- Suppose that $f : (2^N)^N \times 2^{F_2} \to (2^N)^N \sqcup 2^{F_2}$ is a Borel reduction from $E_1 \times E_\infty$ to $E_1 \sqcup E_\infty$.
- Then for all $x \in (2^N)^N$, $f(x, y) \in 2^{F_2}$ for $\mu$-a.e. $y \in 2^{F_2}$.
- By Fubini, there exists $y \in 2^{F_2}$ such that $f(x, y) \in 2^{F_2}$ for $\nu$-a.e. $x \in (2^N)^N$. 

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**Proof.**

- Let $\mu$, $\nu$ be the usual product probability measures on $2^{\mathcal{F}_2}$, $(2^N)_\mathbb{N}$.
- Suppose that $f : (2^N)_\mathbb{N} \times 2^{\mathcal{F}_2} \to (2^N)_\mathbb{N} \uplus 2^{\mathcal{F}_2}$ is a Borel reduction from $E_1 \times E_\infty$ to $E_1 \uplus E_\infty$.
- Then for all $x \in (2^N)_\mathbb{N}$, $f(x, y) \in 2^{\mathcal{F}_2}$ for $\mu$-a.e. $y \in 2^{\mathcal{F}_2}$.
- By Fubini, there exists $y \in 2^{\mathcal{F}_2}$ such that $f(x, y) \in 2^{\mathcal{F}_2}$ for $\nu$-a.e. $x \in (2^N)_\mathbb{N}$.
- But by Kechris, $E_1$ is not $\nu$-a.e. Borel reducible to a countable Borel equivalence relation.
Two Borel reductions

Lemma

\[ \cong \leq_B \approx_F. \]
## Two Borel reductions

**Lemma**
\[ \cong \leq_B \cong_F. \]

**Proof.**
If \( G, H \) are any groups, then

\( G \wr Z \cong H \wr Z \) iff \( G \cong H \).

\( G \wr Z \) has no nontrivial finite normal subgroups. Thus \( G \mapsto G \wr Z \) is a Borel reduction from \( \cong \leq_B \cong_F \).
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If \( G, H \) are any groups, then

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Thus \( G \mapsto G \wr \mathbb{Z} \) is a Borel reduction from \( \cong \) to \( \approx_F \).
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If \( G, H \) are any groups, then

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Thus \( G \leftrightarrow G \text{ wr } \mathbb{Z} \) is a Borel reduction from \( \cong \) to \( \cong_F \).

Lemma

\( E_1 \leq_B \cong. \)
Two Borel reductions

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If \( G, H \) are any groups, then

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- \( G \text{ wr } \mathbb{Z} \) has no nontrivial finite normal subgroups.

Thus \( G \mapsto G \text{ wr } \mathbb{Z} \) is a Borel reduction from \( \cong \) to \( \approx_F \).

Lemma
\[ E_1 \leq_B \approx. \]

Proof.

There exists a Borel reduction \( s \mapsto G_s \), where \( G_s \) is a suitable central extension of a fixed Tarski monster \( M \).
The quasi-equality relation

**Definition**

The f.g. groups $G$, $H$ are **quasi-equal**, written $G \simeq H$, iff there exist finite normal subgroups $N \trianglelefteq G$ and $M \trianglelefteq H$ such that $G/N = H/M$ as marked groups.

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The quasi-equality relation on $\mathcal{G}$ corresponds to the following equivalence relation on the corresponding subspace $\mathcal{N}$ of normal subgroups of $\mathbb{F}_\infty$.

Definition

If $A, B \in \mathcal{N}$, then $A$ and $B$ are said to be **quasi-equal**, written $A \simeq B$, iff $[A : A \cap B], [B : A \cap B] < \infty$. 
The quasi-equality relation continued

Theorem (S.T.)

\[ \sim \text{ is Borel bireducible with } E_1. \]
The quasi-equality relation continued

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Note that if \( G, H \in \mathcal{G} \), then

\[ G \equiv_F H \iff \exists \pi \in \text{Aut}_f(\mathcal{F}_\infty) \quad \pi \cdot [G] = [H], \]

where \([G]\) is the corresponding quasi-equality class.
The quasi-equality relation continued

**Theorem (S.T.)**

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- Note that if \( G, H \in \mathcal{G} \), then
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  where \([G]\) is the corresponding quasi-equality class.
- Thus \( \sim_F \) is an “extension” of \( E_1 \) by \( E_\infty \).
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- Note that if \(G, H \in \mathcal{G}\), then

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where \([G]\) is the corresponding quasi-equality class.

- Thus \(\sim_F\) is an “extension” of \(E_1\) by \(E_\infty\).

**Conjecture**

\(\sim_F\) is a “nonsplit extension”; i.e. \(E_1 \times E_\infty <_B \sim_F\).
The quasi-equality relation continued

Theorem (S.T.)

\( \sim \) is Borel bireducible with \( E_1 \).

- Note that if \( G, H \in \mathcal{G} \), then
  \[
  G \approx_F H \quad \text{iff} \quad \exists \pi \in \text{Aut}_f(F_\infty) \quad \pi \cdot [G] = [H],
  \]
  where \([G]\) is the corresponding quasi-equality class.
- Thus \( \approx_F \) is an "extension" of \( E_1 \) by \( E_\infty \).

Conjecture

\( \approx_F \) is a "nonsplit extension"; i.e. \( E_1 \times E_\infty <_B \approx_F \).

Conjecture

\( E_K_\sigma \) is not an extension of \( E_1 \) by \( E_\infty \).
The quasi-equality relation is hypersmooth

For each $A \in \mathbb{N}$, let $G_A = F_\infty / A$.

Then we can define a characteristic subgroup $\Delta^+ (G_A) = \{ g \in G_A \mid$ the normal closure $\langle g G_A \rangle$ is finite $\}$.

Let $A^+ \in \mathbb{N}$ be the corresponding normal subgroup such that $F_\infty / A^+ = G_A / \Delta^+ (G_A)$.

Fix a linear ordering $\prec$ of $F_\infty$ and define the Borel map $U : \mathbb{N} \to \mathbb{N}$ by:

- If $\Delta^+ (G_A) = 1$, then $A^+ = A$.
- Otherwise, let $g \in A^+ \setminus A$ be the $\prec$-least element and let $U(A)$ be the normal closure of $A \cup \{g\}$ in $F_\infty$.

If $A, B \in \mathbb{N}$, then $A \simeq B$ iff there exist $n, m \geq 1$ such that $U_n(A) = U_m(B)$.
The quasi-equality relation is hypersmooth

- For each \( A \in \mathcal{N} \), let \( G_A = F_\infty / A \).
The quasi-equality relation is hypersmooth

- For each $A \in \mathcal{N}$, let $G_A = F_\infty / A$.
- Then we can define a characteristic subgroup

$$\Delta^+(G_A) = \{ g \in G_A \mid \text{the normal closure } \langle g^{G_A} \rangle \text{ is finite} \}.$$

If $\Delta^+(G_A) = 1$, then $A^+ = A$.
Otherwise, let $g \in A^+ \setminus A$ be the $\prec$-least element and let $U(A)$ be the normal closure of $A \cup \{g\}$ in $F_\infty$.

If $A, B \in \mathcal{N}$, then $A \approx B$ iff there exist $n, m \geq 1$ such that $U_n(A) = U_m(B)$. 

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The quasi-equality relation is hypersmooth

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  - If $\Delta^+(G_A) = 1$, then $A^+ = A$.
  - Otherwise, let $g \in A^+ \setminus A$ be the $\prec$-least element and let $U(A)$ be the normal closure of $A \cup \{g\}$ in $F$.
- If $A, B \in \mathcal{N}$, then $A \simeq B$ iff there exist $n, m \geq 1$ such that
  \[ U^n(A) = U^m(B) . \]
The f.g. groups $G_1$, $G_2$ are **virtually isomorphic**, written $G_1 \approx_v G_2$, iff there exist subgroups $N_i \leq H_i \leq G_i$ such that:

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- $N_1, N_2$ are finite normal subgroups of $H_1, H_2$ respectively.
- $H_1/N_1 \cong H_2/N_2$. 

**Theorem (S.T.)**

$E_1 \times E_\infty \leq B \approx_v \leq B E K \sigma$.

**Conjecture**

$\approx_v \sim B \approx_f$. 

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The virtual isomorphism relation

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**Theorem (S.T.)**

$E_1 \times E_\infty \leq_B \approx_V \leq_B E_{K_\sigma}$. 
The virtual isomorphism relation

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The f.g. groups $G_1$, $G_2$ are **virtually isomorphic**, written $G_1 \approx_V G_2$, iff there exist subgroups $N_i \leq H_i \leq G_i$ such that:

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$E_1 \times E_\infty \leq_B \approx_V \leq_B E_{K_\sigma}$.

**Conjecture**

$\approx_V \sim_B \approx_F$.

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$E_{k_\sigma}$ \iff quasi-isometry for f.g. groups

virtual isomorphism for f.g. groups

$E_1 \times E_\infty$

$E_1 \leadsto E_\infty = \text{isomorphism for f.g. groups}$

$E_0$

$id_\mathbb{R}$