THE BI-EMBEDDABILITY RELATION FOR COUNTABLE ABELIAN GROUPS

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Abstract. We analyze the complexity of the bi-embeddability relations for countable torsion-free abelian groups and for countable torsion abelian groups.

1. Introduction

In this paper, we will analyze the complexity of the bi-embeddability relations on the standard Borel spaces of countable torsion-free abelian groups and countable torsion abelian groups.

Theorem 1.1. *The embeddability relation $\sqsubseteq_{TFA}$ on the space of countable torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ quasi-order.*

Corollary 1.2. *The bi-embeddability relation $\equiv_{TFA}$ on the space of countable torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.*

It is well known that there exist many analytic equivalence relations which are not Borel reducible to the isomorphism relation on any class of countable structures. (For example, see Hjorth [11].) In particular, it follows that the isomorphism relation $\cong_{TFA}$ on the space of countable torsion-free abelian groups is not a complete $\mathbf{\Sigma}^1_1$ equivalence relation; and thus $\equiv_{TFA}$ is strictly more complex than $\cong_{TFA}$ with respect to Borel reducibility. On the other hand, it turns out that the situation is quite different for the space of countable torsion abelian groups.

Theorem 1.3. *The isomorphism $\cong_{TA}$ and bi-embeddability $\equiv_{TA}$ relations on the space of countable torsion abelian groups are incomparable with respect to Borel reducibility.*

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However, it has to be admitted that Theorem 1.3 is somewhat counterintuitive: as we will see in Sections 4 and 5, the bi-embeddability relation $\equiv_{TA}$ has a strictly simpler complete invariant than the isomorphism relation $\cong_{TA}$. On the other hand, under a relatively mild large cardinal assumption, we obtain the intuitively correct result if we replace Borel reducibility by $\mathbf{\Delta}^1_2$ reducibility.

Throughout this paper, we will write (RC) to indicate that the proof of a given result makes use of the assumption that a Ramsey cardinal exists.

Theorem 1.4 (RC)**.** *The isomorphism relation $\cong_{TA}$ on the space of countable torsion abelian groups is strictly more complex with respect to $\mathbf{\Delta}^1_2$ reducibility than the bi-embeddability relation $\equiv_{TA}$.*

This paper is organised as follows. In Section 2, we will recall some basic notions and results concerning Borel and $\mathbf{\Delta}^1_2$ reductions between analytic equivalence relations and quasi-orders; and we will discuss the absoluteness of these notions. In Section 3, adapting the techniques of Louveau-Rosendal [17] and Downey-Montalban [3], we will prove that embeddability relation $\sqsubseteq_{TFA}$ on the space of countable torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ quasi-order. In Section 4, we will discuss the Ulm factor analysis of the isomorphism $\cong_p$ and the bi-embeddability $\equiv_p$ relations on the space of countable abelian p-groups; and we will prove the analogs of Theorems 1.3 and 1.4 for $\cong_p$ and $\equiv_p$. In Section 5, we will use the theory of pinned names to prove Theorems 1.3 and 1.4. Finally, in Appendix A, we will explain how to derive our Ulm factor analysis of the bi-embeddability $\equiv_p$ relation from the equivalent result in Barwise-Eklof [1].

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2. Preliminaries

In this section, we will recall some basic notions and results concerning Borel and $\mathbf{\Delta}^1_2$ reductions between analytic equivalence relations and quasi-orders; and we will discuss the absoluteness of these notions.

2.1. **Reductions and homomorphisms.** Suppose that R, S are binary relations on the Polish spaces X, Z. Then a map $f : X \to Z$ is said to be a *homomorphism* from R to S if $x, y \in X$,

$$x \, R \, y \quad \Longrightarrow \quad f(x) \, S \, f(y).$$

If f satisfies the stronger property that for all $x, y \in X$,

$$x \, R \, y \quad \Longleftrightarrow \quad f(x) \, S \, f(y),$$

then f is said to be a *reduction* from R to S. In this paper, we will be interested in the cases when R, S are either analytic equivalence relations or analytic quasi-orders, and when the map f is either Borel or $\mathbf{\Delta}^1_2$.

2.2. **Complete analytic quasi-orders and equivalence relations.** An analytic quasi-order R on a Polish space X is said to be a *complete* $\mathbf{\Sigma}^1_1$ *quasi-order* if whenever S is an analytic quasi-order on a Polish space Y, then S is Borel reducible to R. Similarly, an analytic equivalence relation E on a Polish space X is said to be *complete* $\mathbf{\Sigma}^1_1$ if whenever F is an analytic equivalence relation on a Polish space Y, then F is Borel reducible to E. For example, if R is a complete $\mathbf{\Sigma}^1_1$ quasi-order on a Polish space X, then the analytic equivalence relation E_R on X, defined by

$$x \, E_R \, z \quad \Longleftrightarrow \quad x \, R \, z \text{ and } z \, R \, x,$$

is complete $\mathbf{\Sigma}^1_1$. In fact, by Louveau-Rosendal [17, Proposition 1.5], every complete $\mathbf{\Sigma}^1_1$ equivalence relation can be obtained in this fashion from a complete $\mathbf{\Sigma}^1_1$ quasi-order. In the rest of this subsection, following Louveau-Rosendal [17, Section 2], we will define the complete $\mathbf{\Sigma}^1_1$ quasi-order $\leq_{max}$.

If X is any set, then $X^{<\omega}$ denotes the set of finite sequences of elements of X; and if $s \in X^{<\omega}$, then $|s|$ denotes the length of the sequence s. If Y is a second set, then we will identify $(X \times Y)^{<\omega}$ with the set of pairs $(s, t) \in X^{<\omega} \times Y^{<\omega}$ of equal length $|s| = |t|$. Let $\leq$ be the partial order on $\omega^{<\omega}$ defined by

$$s \leq t \quad \Longleftrightarrow \quad |s| = |t| \text{ and } s(i) \leq t(i) \text{ for all } i < |s|.$$

Suppose that $T \subseteq (2 \times \omega)^{<\omega}$ is a (set-theoretic) tree; and for each $s \in \omega^{<\omega}$, let

$$T(s) = \{ u \in 2^{<\omega} \mid |u| = |s| \text{ and } (u, s) \in T \}.$$

Then T is said to be *normal* if whenever $s \leq t$, then $T(s) \subseteq T(t)$.

Definition 2.1. Let $\mathcal{NT}$ be the standard Borel space of normal trees on $2 \times \omega$. Then $\leq_{max}$ is the $\mathbf{\Sigma}^1_1$ quasi-order on $\mathcal{NT}$ defined by $S \leq_{max} T$ if and only if there exists a Lipschitz map $f : \omega^{<\omega} \to \omega^{<\omega}$ such that $S(s) \subseteq T(f(s))$ for all $s \in \omega^{<\omega}$.

Here $f : \omega^{<\omega} \to \omega^{<\omega}$ is said to be *Lipschitz* if there exists a map $f^* : \omega^{<\omega} \times \omega \to \omega$ such that $f(\emptyset) = \emptyset$ and $f(s ^\frown n) = f(s) ^\frown f^*(s, n)$.

Theorem 2.2 (Louveau-Rosendal [17])**.** *$\leq_{max}$ is a complete $\mathbf{\Sigma}^1_1$ quasi-order.*

Thus, in order to prove Theorem 1.1, it is enough to show that $\leq_{max}$ is Borel reducible to $\sqsubseteq_{TFA}$. The proof of Theorem 1.1 also makes use of the following observation of Louveau-Rosendal [17, Remark 2.6.2].

Lemma 2.3. *If T, $U \in \mathcal{NT}$ and $T \leq_{max} U$, then there exists an* injective *Lipschitz map $f : \omega^{<\omega} \to \omega^{<\omega}$ such that $T(s) \subseteq U(f(s))$ for all $s \in \omega^{<\omega}$.*

2.3. **Absoluteness.** Let V be a fixed base universe of set theory and let $\mathbb{P}$ be a notion of forcing. Then we will write $V^{\mathbb{P}}$ for the corresponding generic extension when we do not wish to specify the generic filter $G \subseteq \mathbb{P}$. If R is a projective relation on the Polish space X, then $X^{V^{\mathbb{P}}}$, $R^{V^{\mathbb{P}}}$ will denote the sets obtained by applying the definitions of X, R within $V^{\mathbb{P}}$. In particular, suppose that E is an analytic equivalence relation on the Polish space X. Then the Shoenfield Absoluteness Theorem [13, Theorem 25.20] implies that $X^{V^{\mathbb{P}}} \cap V = X$ and $E^{V^{\mathbb{P}}} \cap V = E$, that $E^{V^{\mathbb{P}}}$ is an analytic equivalence relation on $X^{V^{\mathbb{P}}}$, and that the following result holds.

Theorem 2.4. *If E, F are analytic equivalence relations on the Polish spaces X, Y and $\theta : X \to Y$ is a Borel reduction from E to F, then $\theta^{V^{\mathbb{P}}}$ is a Borel reduction from $E^{V^{\mathbb{P}}}$ to $F^{V^{\mathbb{P}}}$.*

Next suppose that $\theta : X \to Y$ is a $\mathbf{\Delta}^1_2$ reduction from E to F; say,

$$\theta(x) = y \iff R(x, y) \iff S(x, y),$$

for all $x \in X$ and $y \in Y$, where R is $\mathbf{\Sigma}^1_2$ and S is $\mathbf{\Pi}^1_2$. Then, without further assumptions on V and $\mathbb{P}$, it is possible that $R^{V^{\mathbb{P}}} \subsetneq S^{V^{\mathbb{P}}}$, that $R^{V^{\mathbb{P}}}$ only defines a partial function from $X^{V^{\mathbb{P}}}$ to $Y^{V^{\mathbb{P}}}$, and that $S^{V^{\mathbb{P}}}$ does not define a function. However, it is easily checked that all of the relevant properties of R, S can be expressed by $\mathbf{\Pi}^1_3$ statements. Thus the following result is a consequence of the Martin-Solovay Absoluteness Theorem [18].

Theorem 2.5. *Suppose that κ is a Ramsey cardinal and that $|\mathbb{P}| < \kappa$. If E, F are analytic equivalence relations on the Polish spaces X, Y and $\theta : X \to Y$ is a $\mathbf{\Delta}^1_2$ reduction from E to F, then $\theta^{V^{\mathbb{P}}}$ is a $\mathbf{\Delta}^1_2$ reduction from $E^{V^{\mathbb{P}}}$ to $F^{V^{\mathbb{P}}}$.*

2.4. **Some miscellaneous notation and conventions.** Throughout this paper, C_n will denote the cyclic group of order n. Also if A is an abelian group and $\ell \in \omega \cup \{ \omega \}$, then $A^{(\ell)}$ will denote the direct sum of ℓ copies of A. As expected, if $\ell = 0$, then $A^{(0)} = 0$ is the trivial abelian group.

Throughout this paper, "countable" will always mean "countably infinite" unless we explicitly write "countable (possibly finite)". The set of natural numbers will be denoted by ω, and $\omega^+ = \{ n \in \omega \mid n \geq 1 \}$.

3. Embeddability of countable torsion-free abelian groups

In this section, adapting the techniques of Louveau-Rosendal [17] and Downey-Montalban [3], we will prove that the complete $\mathbf{\Sigma}^1_1$ quasi-order $\leq_{max}$ is Borel reducible to the embeddability relation $\sqsubseteq_{TFA}$ on the space of countable torsion-free abelian groups. It follows that $\sqsubseteq_{TFA}$ is a complete $\mathbf{\Sigma}^1_1$ quasi-order and that the bi-embeddability relation $\equiv_{TFA}$ on the space of countable torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

Recall that a graph T is said to be a *combinatorial tree* if T is connected and acyclic; and that a *rooted combinatorial tree* is a combinatorial tree, together with a distinguished vertex t_0 which is called its *root*. If (T, t_0) is a rooted combinatorial tree, then we can define a natural partial order $\preceq$ on T by setting $s \preceq t$ if the unique path from t_0 to t contains s. For each $t \in T$, the *height* $|t|$ of t is the length of the unique path from t_0 to t; and for each $t \in T \setminus \{ t_0 \}$, the immediate $\preceq$-predecessor of t is denoted by t^-. Similarly, if $S \subseteq \omega^{<\omega}$ is a set-theoretic tree and $\emptyset \neq s \in S$ is a nonempty sequence, then $s^- = s \restriction (|s| - 1)$ is the sequence obtained by removing the last element of s.

Let $\mathcal{CT}$ be the standard Borel space of countable combinatorial rooted trees and let $\sqsubseteq_{RCT}$ be the (graph theoretic) embeddability relation on $\mathcal{CT}$. Then, following Louveau-Rosendal [17], for each normal tree $T \in \mathcal{NT}$, we will define a corresponding combinatorial rooted tree $G_T \in \mathcal{T}$ as follows. First, let $\{\theta(n) \mid n \in \omega\}$ be the enumeration of $2^{<\omega}$ induced by the lexicographical ordering. Next, let G_0 be the combinatorial rooted tree with vertex set

$$\omega^{<\omega} \sqcup \{ s^* \mid s \in \omega^{<\omega} \smallsetminus \{\emptyset\} \}$$

and edge set

$$\{ \{ s, s^* \} \mid s \in \omega^{<\omega} \smallsetminus \{\emptyset\} \} \sqcup \{ \{ s^-, s^* \} \mid s \in \omega^{<\omega} \smallsetminus \{\emptyset\} \}$$

and root $\emptyset$. Finally, for each $(u, s) \in T$, we add vertices (u, s, x), where x is either 0^k or $0^{2\theta(u)+2} \frown 1 \frown 0^k$ for some $k \in \omega$; and we link each vertex (u, s, x) with $x \neq \emptyset$ to the vertex (u, s, x^-), and we link each vertex $(u, s, \emptyset)$ to s.

Remark 3.1. Although we will not directly use this result, it is perhaps worth mentioning that, by Louveau-Rosendal [17, Theorem 3.1], the map $T \mapsto G_T$ is a Borel reduction from $\leq_{max}$ to $\sqsubseteq_{RCT}$.

The degree of each vertex $v \in G_T$ is easily computed: each vertex $s \in \omega^{<\omega}$ has infinite degree; each vertex $(u, s, 0^{2\theta(u)+2})$, for $(u, s) \in T$, has degree 3; and all other vertices have degree 2.

Definition 3.2. Let G be a rooted combinatorial tree such that every vertex has degree 2, 3 or ω and let E be the edge relation on G. Let V be the vector space over $\mathbb{Q}$ with basis G. Let $\{ p_n \mid n \in \omega \}$ be the set of prime natural numbers. Then $A(G)$ is the additive subgroup of V generated by the elements of the following form:

- $t/p^k_{4|t|}$ for $k \in \omega$ and $t \in G$ of degree ω;
- $t/p^k_{4|t|+1}$ for $k \in \omega$ and $t \in G$ of degree 2;
- $t/p^k_{4|t|+2}$ for $k \in \omega$ and $t \in G$ of degree 3;
- $(t + u)/p^k_{4|t|+3}$ for $k \in \omega$ and $\{ t, u \} \in E$ with $t = u^-$.

The remainder of this section will be devoted to the proof of the following result.

Theorem 3.3. *The map $T \mapsto A(G_T)$ is a Borel reduction from $\leq_{max}$ to $\sqsubseteq_{TFA}$.*

Let T, $U \in \mathcal{NT}$. First suppose that $T \leq_{max} U$. Then, by Lemma 2.3, there always exists an injective Lipschitz map $f : \omega^{<\omega} \to \omega^{<\omega}$ such that $T(s) \subseteq U(f(s))$ for all $s \in \omega^{<\omega}$. Following the proof of Louveau-Rosendal [17, Theorem 3.1], we can extend f to a map $\varphi : G_T \to G_U$ as follows. First for each $s \in \omega^{<\omega}$, let $\varphi(s) = f(s)$ and let $\varphi(s^*) = f(s)^*$. Next, if $(u, s) \in T$, then $(u, f(s)) \in U$ and so we can define $\varphi(u, s, x) = (u, f(s), x)$. It is easily checked that $\varphi : G_T \to G_U$ is a degree-preserving embedding of rooted combinatorial trees, and it follows that φ extends to an embedding $\varphi : A(G_T) \to A(G_U)$.

Next suppose that $\varphi : A(G_T) \to A(G_U)$ is an embedding. Recall that $A(G_U)$ is an additive subgroup of the vector space $V = \bigoplus_{t \in G_U} \mathbb{Q}\, t$. For each element $v = \sum_{t \in G_U} q_t\, t \in V$, let $\operatorname{supp}(v) = \{ t \in G_U \mid q_t \neq 0 \}$.

Definition 3.4.

- For each vertex $t \in G_T$, let $S_t = \operatorname{supp} \varphi(t)$.
- For each edge $e = \{ t, u \}$ of G_T, let $E_e = \operatorname{supp} \varphi(t + u)$.

The next two lemmas are straightforward variants of the corresponding results in Downey-Montalban [3, Section 2].

Lemma 3.5. *If $t \in G_T$, then $S_t \subseteq \{ r \in G_U \mid |r| = |t|$ and $\deg(r) = \deg(t) \}$.*

Sketch proof. By definition, $\varphi(t) = \sum_{u \in S_t} q_u u$ for some $q_u \in \mathbb{Q} \smallsetminus \{ 0 \}$. Suppose, for example, that $\deg(t) = 2$. Then t is divisible in $A(G_T)$ by $p^k_{4|t|+1}$ for all $k \in \omega$, and so the same is true of $\varphi(t)$ in $A(G_U)$. Arguing exactly as in the proof of Downey-Montalban [3, Lemma 2.3], we see that $|u| = |t|$ and $\deg(u) = 2$ for each $u \in S_t$. □

In particular, it follows that $S_\emptyset = \{ \emptyset \}$.

Lemma 3.6. *Let $e = \{ t, u \}$ be an edge of G_T with $t = u^-$.*

(i) *$E_e = S_t \cup S_u$.*

(ii) *For all $r \in S_t$, there exists $s \in S_u$ such that $\{ r, s \}$ is an edge of G_U.*

Sketch proof. Lemma 3.5 implies that $\operatorname{supp} \varphi(t) \cap \operatorname{supp} \varphi(u) = \emptyset$ and it follows that

$$E_e = \operatorname{supp} \varphi(t + u) = \operatorname{supp} \varphi(t) \cup \operatorname{supp} \varphi(u) = S_t \cup S_u.$$

Also $\varphi(t + u)$ is divisible in $A(G_U)$ by $p^k_{4|t|+3}$ for all $k \in \omega$. Arguing exactly as in the proof of Downey-Montalban [3, Lemma 2.4], we see that for each $r \in S_t$, there exists $s \in S_u$ such that $\{ r, s \}$ is an edge of G_U. □

Lemma 3.7. *There exists a function $f : G_T \to G_U$ such that:*

- $f(t) \in S_t$;
- *if* $\{ t, u \}$ *is an edge of* G_T, *then* $\{ f(t), f(u) \}$ *is an edge of* G_U.

Remark 3.8. Note that we do *not* require that f should be an injection.

Proof of Lemma 3.7. We will define $f(t)$ by induction on $|t|$. First we set $f(\emptyset) = \emptyset$. Next suppose that $f(t)$ has been defined and that $f(t) \in S_t$. Let $e = \{ t, u \}$ be an edge of G_T with $t = u^-$. Then, by Lemma 3.6, there exists $s \in S_u$ such that $(f(t), s)$ is an edge of G_U, and so we can set $f(u) = s$. □

Let $f : G_T \to G_U$ be the function given by Lemma 3.7. Then, applying Lemma 3.5, since $f(t) \in S_t$ for every $t \in G_T$, it follows that f is height-preserving and degree-preserving. In particular, it follows that $f[\omega^\omega] \subseteq \omega^\omega$. We claim that $f \restriction \omega^\omega$ is a Lipschitz map. To see this, suppose that $r \in \omega^\omega$ and that $s = r ^\frown n$ for some $n \in \omega$. Then $f(s^*)$ is an immediate successor of $f(r)$ and an immediate predecessor of $f(s) \in \omega^\omega$. It follows easily that there exists $m \in \omega$ such that $f(s) = f(r) ^\frown m$.

We claim that f witnesses that $T \leq_{max} U$. To see this, suppose that $(u, s) \in T$. Then, in G_T, the vertex $s \in \omega^\omega$ is below the vertex $(u, s, 0^{2\theta(u)+2})$, which is of degree 3 and height $|s| + 2\theta(u) + 3$. It follows that the vertex $f(s) \in \omega^\omega$ is below a vertex $v \in G_U$ of degree 3 and height

$$|s| + 2\theta(u) + 3 = |f(s)| + 2\theta(u) + 3,$$

and the only possibility is that $v = (u, f(s), 0^{2\theta(u)+2})$. Thus $(u, f(s)) \in U$, as required. This completes the proof of Theorem 3.3.

4. Bi-embeddability of countable abelian p-groups

Let $\mathcal{A}_p$ be the standard Borel space of countable abelian p-groups. Let $\cong_p$ be the isomorphism relation on $\mathcal{A}_p$ and let $\equiv_p$ be the bi-embeddability relation on $\mathcal{A}_p$. Then it is clear that $\cong_p$ and $\equiv_p$ are analytic equivalence relations. By Friedman-Stanley [5], the isomorphism relation $\cong_p$ is non-Borel but is not Borel complete. In [1], Barwise-Eklof found a complete set of invariants for the bi-embeddability relation $\equiv_p$, which showed that there are exactly ω_1 countable abelian p-groups up to bi-embeddability. It follows that the bi-embeddability relation $\equiv_p$ is also non-Borel and not Borel complete. In this section, we will prove the analogs of Theorems

1.3 and 1.4 for $\cong_p$ and $\equiv_p$. We will begin by recalling the Ulm analysis [21] of the isomorphism relation $\cong_p$ on $\mathcal{A}_p$. (There are two closely related approaches to the Ulm analysis of $\cong_p$; namely, in terms of Ulm factors and in terms of Ulm-Kaplansky invariants. In this paper, we will take the Ulm factor approach.)

Suppose that A is an arbitrary (not necessarily countable) abelian p-group. Then the *α-th Ulm subgroup* A^α is defined inductively by:

- $A^0 = A$;
- $A^{\alpha+1} = \bigcap_{n<\omega} p^n A^\alpha$;
- $A^\delta = \bigcap_{\alpha<\delta} A^\alpha$, if δ is a limit ordinal.

There exists an ordinal $\tau < |A|^+$ such that $A^\tau = A^{\tau+1}$ and the *Ulm length* $\tau(A)$ of A is defined to be the least such ordinal τ. Clearly $A^{\tau(A)}$ is the maximal divisible subgroup of A and so $A^{\tau(A)}$ is isomorphic to a direct sum of κ copies of the quasi-cyclic group $\mathbb{Z}(p^\infty)$ for some cardinal $0 \le \kappa \le |A|$. We define κ to be the *rank* of $A^{\tau(A)}$ and we write $\mathrm{rk}(A^{\tau(A)}) = \kappa$. The abelian p-group A is said to be *reduced* if $A^{\tau(A)} = 0$. For each $\alpha < \tau(A)$, the *αth Ulm factor* of A is the factor group $A_\alpha = A^\alpha / A^{\alpha+1}$. Recall that A can be expressed as the direct sum $A = A^{\tau(A)} \oplus C$ of its maximal divisible subgroup $A^{\tau(A)}$ and a reduced subgroup C. (For example, see Fuchs [7, Theorem 21.3].) Furthermore, it is easily checked that $\tau(A) = \tau(C)$ and that the Ulm factors A_α, C_α are isomorphic for all $\alpha < \tau(A) = \tau(C)$.

Of course, if A is a countable abelian p-group, then $\tau(A)$ is a countable ordinal and $0 \le \mathrm{rk}(A^{\tau(A)}) \le \omega$. In addition, it follows that each Ulm factor A_α is a Σ-cyclic p-group; i.e. is a direct sum of cyclic p-groups. (See Fuchs [8, Section 76].)

Theorem 4.1 (Ulm [21]). *If A and B are countable abelian p-groups, then A is isomorphic to B if and only if the following conditions are satisfied:*

(i) $\tau(A) = \tau(B)$;

(ii) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)})$;

(iii) *for each $\alpha < \tau(A) = \tau(B)$, the Ulm factors A_α and B_α are isomorphic.*

We will next consider the question of which sequences of Σ-cyclic p-groups can be realized as the Ulm factors of a countable abelian p-group. Recall that a Σ-cyclic p-group H is said to be *bounded* if there exists an integer $n \ge 0$ such that $p^n h = 0$ for all $h \in H$. It is well-known that if A is a countable abelian p-group, then each Ulm factor A_α must be unbounded, except possibly for $A_{\tau(A)-1}$, if this factor exists. (For example, see Fuchs [7, Lemma 37.2].) In fact, this is the only constraint on the possible Ulm factors of countable abelian p-groups.

Theorem 4.2 (Zippin [23]). *Suppose that $0 < \tau < \omega_1$ is a nonzero countable ordinal and that $(C_\alpha \mid \alpha < \tau)$ is a sequence of nontrivial countable (possibly finite) Σ-cyclic p-groups. Then the following statements are equivalent:*

(i) *There exists a countable reduced abelian p-group A with $\tau(A) = \tau$ such that $A_\alpha \cong C_\alpha$ for all $\alpha < \tau$.*

(ii) *C_α is unbounded for each α such that $\alpha + 1 < \tau$.*

Remark 4.3. Extending Zippin's Theorem, Fuchs [6] and Kulikov [16] have given necessary and sufficient conditions for a sequence $(C_\alpha \mid \alpha < \tau)$ of abelian p-groups to be realizable as the Ulm sequence of a reduced abelian p-group of cardinality κ, when τ and κ are not assumed to be countable. (See Fuchs [8, Theorem 76.1].) We will make use of a special case of the Fuchs-Kulikov Theorem in Section 5.

Each countable (possibly finite) Σ-cyclic p-group has the form $G = \bigoplus_{n \geq 1} C_{p^n}^{(s_n)}$, where each $s_n \in \omega \cup \{ \omega \}$; and clearly G is determined up to isomorphism by the sequence $t_G = (s_n \mid n \in \omega^+)$. Thus, each countable abelian p-group A is determined up to isomorphism by the complete invariant

$$\tau(A)^\frown (t_{A_\alpha} \mid \alpha < \tau(A))^\frown \operatorname{rk}(A^{\tau(A)}). \tag{4.1}$$

In particular, we obtain the same set of complete invariants (4.1), independently of our choice of the prime p. Unfortunately, the complete invariant (4.1) cannot be computed in a Borel manner. For example, applying the Boundedness Theorem [9, Theorem 1.6.10] for $\mathbf{\Sigma}^1_1$ sets of well-orders, it follows that there does not exist a Borel map $A \mapsto L_A$ from $\mathcal{A}_p$ to the standard Borel space of countable linear orders such that $L_A \cong \tau(A)$. Thus it is natural to ask the following question.

Question 4.4. Are $\cong_p$ and $\cong_q$ Borel bireducible when $p \neq q$ are distinct primes?

Theorem 4.5. *If $p \neq q$ are distinct primes, then $\cong_p$ and $\cong_q$ are $\mathbf{\Delta}^1_2$ bireducible.*

Let $\mathcal{L}$ be the standard Borel space of countable (possibly finite) linear orders; i.e. each $x \in \mathcal{L}$ consists of a linear ordering $<_x$ of $\operatorname{dom}(x) \in \omega \cup \{ \omega \}$. Let Z the standard Borel space of sequences

$$c = x^\frown (\, t_\ell \mid \ell \in \mathrm{dom}(x)\,)^\frown d, \tag{4.2}$$

where $x \in \mathcal{L}$, $d \in \omega \cup \{\,\omega\,\}$, and each $t_\ell : \omega^+ \to \omega \cup \{\,\omega\,\}$. Let $C \subseteq Z$ be the $\mathbf{\Pi}^1_1$ subset consisting of the sequences (4.2) such that:

- $<_x$ is a well-ordering of $\mathrm{dom}(x)$;
- for each $\ell \in \mathrm{dom}(x)$, there exists $n \in \omega^+$ such that $t_\ell(n) \neq 0$;
- if ℓ is not $<_x$-maximal, then $t_\ell(n) \neq 0$ for infinitely many $n \in \omega^+$.

Then each sequence $c \in C$ naturally codes a corresponding complete invariant (4.1), which we will denote by $[\,c\,]$.

While we do not know of any reference where detailed proofs can be found, the following definability results are well-known. (For example, see the comments in Friedman-Stanley [5] and Hjorth-Kechris [12].) For a splendidly elephantine proof[1], the reader might check that the binary relation $I(c, A) \subseteq Z \times \mathcal{A}_p$, defined by

$$[\,c\,] = \tau(A)^\frown (\, t_{A_\alpha} \mid \alpha < \tau(A)\,)^\frown \mathrm{rk}(A^{\tau(A)}),$$

is $\mathbf{\Sigma}^1_2$ and then apply the Kondô Uniformization Theorem [15, Corollary 38.7].

Lemma 4.6. *For each prime p, there exists a $\mathbf{\Delta}^1_2$ map $\theta_p : \mathcal{A}_p \to Z$ such that*

$$[\,\theta_p(A)\,] = \tau(A)^\frown (\, t_{A_\alpha} \mid \alpha < \tau(A)\,)^\frown \mathrm{rk}(A^{\tau(A)}).$$

Lemma 4.7. *For each prime p, there exists a $\mathbf{\Delta}^1_2$ map $\varphi_p : Z \to \mathcal{A}_p$ such that if $c \in C$ and $A = \varphi_p(c)$, then*

$$[\,c\,] = \tau(A)^\frown (\, t_{A_\alpha} \mid \alpha < \tau(A)\,)^\frown \mathrm{rk}(A^{\tau(A)}).$$

Proof of Theorem 4.5. If $p \neq q$ are distinct primes, then $\varphi_q \circ \theta_p$ is a $\mathbf{\Delta}^1_2$ reduction from $\cong_p$ to $\cong_q$. □

We next recall a very useful "tree presentation" approach to countable abelian p-groups, which was first introduced by Crawley-Hales [2]. Let $\mathcal{T}$ be the standard Borel space of infinite trees $T \subseteq \omega^{<\omega}$; and for each $T \in \mathcal{T}$, let $G_p(T)$ be the abelian group generated by the elements $\{\, a_t \mid t \in T\,\}$ subject to the relations:

- $p\, a_t = a_{t^-}$ if $|t| > 0$;

[1] Cf. Neumann's proof [19, p. 347] of Fermat's "Little Theorem".

- $a_\emptyset = 0$.

(Following the example of Rogers [20], we have included the redundant generator $a_\emptyset$ so that our groups $G_p(T)$ are parametrized by trees rather than forests.) Then $G_p(T)$ is a p-group, and we can identify each $G_p(T)$ with a corresponding element of $\mathcal{A}_p$ in such a way that the map $T \overset{\varphi}{\mapsto} G_p(T)$ is Borel. In fact, the following result shows that $\varphi(\mathcal{T})$ intersects every $\cong_p$-class.

Theorem 4.8 (Crawley-Hales [2]). *If A is a countable abelian p-group, then there exists an infinite tree $T \subseteq \omega^{<\omega}$ such that $A \cong G_p(T)$.*

The following result is implicitly contained in Rogers [20].

Theorem 4.9. *For any countable trees S, $T \in \mathcal{T}$ and any two primes p, q,*

$$G_p(S) \cong G_p(T) \iff G_q(S) \cong G_q(T)$$

Sketch proof. By the proof of Rogers [20, Proposition 2], if p is any prime and $T \in \mathcal{T}$, then the Ulm-Kaplansky invariants of the group $G_p(T)$ can be computed from the tree T via a computation which does not depend on the prime p. □

Thus a positive answer to the following question would yield a positive answer to Question 4.4.

Question 4.10. Does there exist a Borel map $A \overset{\psi}{\mapsto} T_A$ from $\mathcal{A}_p$ to $\mathcal{T}$ such that $A \cong G_p(T_A)$?

Next we will consider the bi-embeddability relation $\equiv_p$ on the space $\mathcal{A}_p$ of countable abelian p-groups. As we mentioned earlier, in [1], Barwise-Eklof found a complete set of invariants for the bi-embeddability relation $\equiv_p$. The following result restates their classification theorem in terms of Ulm factors. (In Appendix A, we will explain how to derive Theorem 4.11 from the corresponding result of Barwise-Eklof [1].)

Theorem 4.11. *If A, B are countable abelian p-groups, then A and B are bi-embeddable if and only if either:*

(a) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) = \omega$; *or*
(b) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) < \omega$ *and the following conditions are satisfied:*

(i) $\tau(A) = \tau(B)$;

(ii) *if* $\tau(A) = \tau(B)$ *is a successor ordinal* $\beta + 1$, *then the Ulm factors* A_β *and* B_β *are bi-embeddable.*

Remark 4.12. Suppose that $G = \bigoplus_{n\geq 1} C_{p^n}^{(s_n)}$ and $H = \bigoplus_{n\geq 1} C_{p^n}^{(t_n)}$ are Σ-cyclic p-groups, where each s_n, $t_n \in \omega \cup \{\, \omega \,\}$. Then G and H are bi-embeddable if and only if one of the following mutually exclusive statements holds.

(i) G and H are isomorphic finite p-groups.

(ii) G and H are both infinite bounded Σ-cyclic p-groups and

- $m_G = \max\{\, n \mid s_n = \omega \,\} = \max\{\, n \mid t_n = \omega \,\} = m_H$;
- $s_n = t_n$ for all $n \geq m_G = m_H$.

(iii) G and H are both unbounded.

Of course, if statement (ii) holds, then there are only finitely many $n \geq m_G = m_H$ such that $s_n = t_n > 0$. In particular, there are only countably many countable (possibly finite) Σ-cyclic p-groups up to bi-embeddability; and we see that there are exactly ω_1 countable abelian p-groups up to bi-embeddability.

Notation 4.13. For the remainder of this paper, let $(\, R_{p,m} \mid m \in \omega \,)$ be a sequence listing a set of representatives of the countably many bi-embeddability classes of nontrivial countable (possibly finite) Σ-cyclic p-groups, chosen so that $R_{p,0} = \bigoplus_{n\geq 1} C_{p^n}^{(\omega)}$ is the representative of the class of unbounded groups. (In fact, we might as well choose every $R_{p,m}$ to be an element of the "largest" $\cong_p$-class contained in its $\equiv_p$-class, in the sense that if $R_{p,m} = \bigoplus_{n\geq 1} C_{p^n}^{(s_n)}$, $H = \bigoplus_{n\geq 1} C_{p^n}^{(t_n)}$ and $H \equiv_p R_{p,m}$, then $t_n \leq s_n$ for all $n \geq 1$.)

Once again, it is natural to ask the following question.

Question 4.14. Are $\equiv_p$ and $\equiv_q$ Borel bireducible when p, q are different primes?

Theorem 4.15. *If* p, q *are distinct primes, then* $\equiv_p$ *and* $\equiv_q$ *are* $\mathbf{\Delta}^1_2$ *bireducible.*

Proof. Let θ_p and φ_q be the $\mathbf{\Delta}^1_2$ maps given by Lemmas 4.6 and 4.7. Then clearly $\varphi_q \circ \theta_p$ is a $\mathbf{\Delta}^1_2$ reduction from $\equiv_p$ to $\equiv_q$. □

Next we will prove that $\equiv_p$ and $\cong_p$ are incomparable with respect to Borel reducibility. Of course, the following result implies that $\equiv_p$ is not Borel reducible to $\cong_p$.

Theorem 4.16. *$\equiv_p$ is not Borel reducible to $\cong_{TA}$.*

Proof. It is well-known that every countable abelian p-group embeds into the infinite rank divisible p-group $\mathbb{Z}(p^\infty)^{(\omega)}$. (For example, see Fuchs [7, Theorem 24.1].) Thus $D_\infty = \{ A \in \mathcal{A}_p \mid \mathrm{rk}(A^{\tau(A)}) = \omega \}$ forms a single $\equiv_p$-class. Since every $\cong_{TA}$-class is Borel, it is enough to prove the following result.

Claim 4.17. *D_∞ is a complete analytic subset of $\mathcal{A}_p$.*

In order to see this, first note that, by Feferman [4], we have that

$$G_p(T) \text{ is reduced} \quad \iff \quad T \text{ is well-founded.}$$

(In fact, Feferman [4] only proves the above equivalence for the case when $p = 2$. However, his argument works for an arbitrary prime p.) Next let $T \mapsto G_p(T)^{(\omega)}$ be the Borel function from $\mathcal{T}$ to $\mathcal{A}_p$ which maps each tree T to the direct sum of ω copies of $G_p(T)$. Then we have that

$$G_p(T)^{(\omega)} \in D_\infty \quad \iff \quad T \text{ is not well-founded,}$$

and hence D_∞ is a complete analytic subset of $\mathcal{A}_p$. □

Theorem 4.18. *$\cong_p$ is not Borel reducible to $\equiv_p$.*

Proof. Suppose that $\cong_p$ is Borel reducible to $\equiv_p$. Applying Theorem 2.4, we can suppose that $2^\omega > \omega_1$. But then we immediately reach a contradiction, since there are 2^ω many $\cong_p$-classes but only ω_1 many $\equiv_p$-classes. □

Corollary 4.19. *$\equiv_p$ and $\cong_p$ are incomparable with respect to Borel reducibility.*

On the other hand, $\equiv_p$ is $\mathbf{\Delta}^1_2$ reducible to $\cong_p$ in the following strong sense.

Definition 4.20. Suppose that $E \subseteq F$ are analytic equivalence relations on the Polish space X. If $\theta : X \to X$ is a homomorphism from F to E such that $\theta(x)\, F\, x$ for all $x \in X$, then we say that *θ selects an E-class within each F-class.* (Of course, this implies that θ is a reduction from E to F.)

Theorem 4.21. *There exists a $\mathbf{\Delta}^1_2$ function $\psi_p : \mathcal{A}_p \to \mathcal{A}_p$ such that θ selects an $\cong_p$-class within each $\equiv_p$-class.*

Proof. Applying Lemma 4.6, let $\theta_p : \mathcal{A}_p \to Z$ be a $\mathbf{\Delta}^1_2$ map such that, letting

$$A \overset{\theta_p}{\mapsto} c = x ^\frown (\, t_\ell \mid \ell \in \operatorname{dom}(x)\,) ^\frown d \in C,$$

we have that

$$[\, c\,] = \tau(A) ^\frown (\, t_{A_\alpha} \mid \alpha < \tau(A)\,) ^\frown \operatorname{rk}(A^{\tau(A)}).$$

For each $c = x ^\frown (\, t_\ell \mid \ell \in \operatorname{dom}(x)\,) ^\frown d \in Z$, let $c' = x ^\frown (\, t'_\ell \mid \ell \in \operatorname{dom}(x)\,) ^\frown d \in Z$ be defined as follows.

- If $\ell \in \operatorname{dom}(x)$ is not $<_x$-maximal, then $t'_\ell(n) = \omega$ for all $n \in \omega^+$.
- If $\ell \in \operatorname{dom}(x)$ is $<_x$-maximal, let $H_c = \bigoplus_{n \geq 1} C_{p^n}^{(t_\ell(n))}$ and let $m \in \omega$ be such that $R_{p,m} \equiv_p H_c$. (Here $(\, R_{p,m} \mid m \in \omega\,)$ is our fixed sequence of representatives of the countably many bi-embeddability classes of nontrivial countable (possibly finite) Σ-cyclic p-groups.) Then t'_ℓ is the function such that $R_{p,m} = \bigoplus_{n \geq 1} C_{p^n}^{(t'_\ell(n))}$.

Clearly the map $c \mapsto c'$ is Borel; and if $c \in C$, then $c' \in C$. Finally, applying Lemma 4.7, let $\varphi_p : Z \to \mathcal{A}_p$ be the $\mathbf{\Delta}^1_2$ map such that if $c' \in C$ and $A' = \varphi_p(c')$, then $[\, c'\,] = \tau(A') ^\frown (\, t_{A'_\alpha} \mid \alpha < \tau(A')\,) ^\frown \operatorname{rk}((A')^{\tau(A')})$. Then the composition map, $A \overset{\theta_p}{\mapsto} c \mapsto c' \overset{\varphi_p}{\mapsto} A'$, satisfies our requirements. □

Finally, arguing as in the proof of Theorem 4.18, the following result is an easy consequence of Theorem 2.5.

Theorem 4.22 (*RC*)**.** *$\cong_p$ is not $\mathbf{\Delta}^1_2$ reducible to $\equiv_p$.*

Corollary 4.23 (*RC*)**.** *The isomorphism relation $\cong_p$ is strictly more complex with respect to $\mathbf{\Delta}^1_2$ reducibility than the bi-embeddability relation $\equiv_p$.*

5. Bi-embeddability of countable torsion abelian groups

In this section, we will prove Theorems 1.3 and 1.4. Clearly Theorem 4.16 implies that $\equiv_{TA}$ is not Borel reducible to $\cong_{TA}$. Hence, in order to prove Theorem 1.3, it is enough to show that $\cong_{TA}$ is not Borel reducible to $\equiv_{TA}$. In fact, we will prove the following stronger result.

Theorem 5.1. *$\cong_p$ is not Borel reducible to $\equiv_{TA}$.*

Similarly, Theorem 1.4 is an immediate consequence of the following two results.

Theorem 5.2 (*RC*). *$\cong_p$ is not $\mathbf{\Delta}^1_2$ reducible to $\equiv_{TA}$.*

Theorem 5.3. *There exists a $\mathbf{\Delta}^1_2$ function which selects an $\cong_{TA}$-class within each $\equiv_{TA}$-class.*

Proof of Theorem 5.3. Let P be the set of prime numbers. Recall that if A is a countable torsion abelian group, then $A = \bigoplus_{p \in P} A_p$ decomposes as the direct sum of its (possibly finite) p-primary components $A_p = \{ a \in A \mid (\exists n \geq 0)\, p^n a = 0 \}$. Furthermore, if $B = \bigoplus_{p \in P} B_p$ is a second countable torsion abelian group, then it is clear that:

- A and B are isomorphic if and only if for every prime p, the (possibly finite) countable abelian p-groups A_p and B_p are isomorphic.
- A and B are bi-embeddable if and only if for every prime p, the (possibly finite) countable abelian p-groups A_p and B_p are bi-embeddable.

Applying Theorem 4.21, for each prime p, let $\psi_p : \mathcal{A}_p \to \mathcal{A}_p$ be a $\mathbf{\Delta}^1_2$ function which selects an $\cong_p$-class within each $\equiv_p$-class. Then $A \mapsto \bigoplus_{p \in P} \psi_p(A_p)$ is a $\mathbf{\Delta}^1_2$ function which selects an $\cong_{TA}$-class within each $\equiv_{TA}$-class. $\square$

The remainder of this section will be devoted to the proofs of Theorems 5.1 and 5.2. First it is necessary to recall some of the basic theory of pinned names. The notion of a pinned name was first abstracted by Kanovei-Reeken [14] from an argument in Hjorth [10, Section 5]. More recently, Zapletal [22] has developed an extensive theory which has uncovered completely unexpected connections between the theory of analytic equivalence relations and other areas of set theory (such as the Singular Cardinal Hypothesis).

Until further notice, we will fix a notion of forcing $\mathbb{P}$ and an analytic equivalence relation E on a Polish space X. Suppose that σ is a $\mathbb{P}$-name for an element of X; i.e. that $\Vdash_{\mathbb{P}} \sigma \in X^{V^{\mathbb{P}}}$. Then σ_{left} and σ_{right} are the $(\mathbb{P} \times \mathbb{P})$-names such that if $G \times H \subseteq (\mathbb{P} \times \mathbb{P})$ is a generic filter, then $\sigma_{\text{left}}[G \times H] = \sigma[G]$ and $\sigma_{\text{right}}[G \times H] = \sigma[H]$.

Definition 5.4. If σ is a $\mathbb{P}$-name for an element of X, then σ is *E-pinned* if

$$\Vdash_{\mathbb{P} \times \mathbb{P}} \ \sigma_{\text{left}}\, E\, \sigma_{\text{right}}$$

Let $X(\mathbb{P}, E)$ be the proper class of all E-pinned $\mathbb{P}$-names. Then we can regard X as a subset of $X(\mathbb{P}, E)$ by identifying each $x \in X$ with the canonical $\mathbb{P}$-name

$\check{x}$ such that $\check{x}[G] = x$ for every generic filter $G \subseteq \mathbb{P}$; and we can extend E to an equivalence relation on $X(\mathbb{P}, E)$ by defining

$$\sigma \, E \, \sigma' \quad \Longleftrightarrow \quad \Vdash_{\mathbb{P}\times\mathbb{P}} \; \sigma_{\text{left}} \, E \, \sigma'_{\text{right}}$$

Definition 5.5. $\lambda_{\mathbb{P}}(E)$ is the number of E-pinned $\mathbb{P}$-names up to E-equivalence.

Theorem 5.6. *If E, F are analytic equivalence relations and E is Borel reducible to F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.*

Proof. Suppose that E, F are analytic equivalence relations on the Polish spaces X, Y and that $\theta : X \to Y$ is a Borel reduction from E to F; say, R is a Borel relation such that for all $x \in X$ and $y \in Y$,

$$\theta(x) = y \quad \Longleftrightarrow \quad R(x, y).$$

Applying Theorem 2.4, if σ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \sigma \in X$, then

$$\Vdash_{\mathbb{P}} (\, \exists y \in Y \,) \, R(\sigma, y);$$

and hence there exists a $\mathbb{P}$-name τ_σ such that

$$\Vdash_{\mathbb{P}} \tau_\sigma \in Y \;\wedge\; R(\sigma, \tau_\sigma).$$

Furthermore, Theorem 2.4 implies that if $\sigma \in X(\mathbb{P}, E)$ is an E-pinned $\mathbb{P}$-name, then τ_σ is an F-pinned $\mathbb{P}$-name; and that if σ, $\sigma' \in X(\mathbb{P}, E)$, then

$$\sigma \, E \, \sigma' \quad \Longleftrightarrow \quad \tau_\sigma \, F \, \tau_{\sigma'}$$

The result follows. □

Similarly, applying Theorem 2.5, we obtain the following result.

Theorem 5.7. *Suppose that κ is a Ramsey cardinal and that $|\mathbb{P}| < \kappa$. If E, F are analytic equivalence relations and E is $\boldsymbol{\Delta}^1_2$ reducible to F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.*

For the remainder of this section, let $\mathbb{P}$ be the notion of forcing consisting of all finite injective partial functions $p : \omega \to \omega_1$. (Thus if $G \subseteq \mathbb{P}$ is a generic filter, then $g = \bigcup G \in V^{\mathbb{P}}$ is a bijection in $V^{\mathbb{P}}$ between ω and ω_1^V.) Then Theorems 5.1 and 5.2 follow from Theorem 5.7 and the following two results.

Proposition 5.8. $\lambda_{\mathbb{P}}(\cong_p) = 2^{\omega_1}$.

Proposition 5.9. $\lambda_{\mathbb{P}}(\equiv_{TA}) = \omega_2^{\omega}$.

For example, suppose that θ is a $\mathbf{\Delta}_2^1$ reduction from $\cong_p$ to $\equiv_{TA}$ and that κ is a Ramsey cardinal. Recall that if $\mathbb{P}'$ is any notion of forcing such that $|\mathbb{P}'| < \kappa$, then κ remains a Ramsey cardinal in $V^{\mathbb{P}'}$. (For example, see Jech [13, Theorem 21.2].) Hence, applying Theorem 2.5, we can suppose that $2^{\omega_1} > \omega_2^{\omega}$. But then

$$\lambda_{\mathbb{P}}(\cong_p) = 2^{\omega_1} > \lambda_{\mathbb{P}}(\equiv_{TA}) = \omega_2^{\omega},$$

which contradicts Theorem 5.7.

In the proofs of Propositions 5.8 and 5.9, we will make use of the following special case of the Fuchs-Kulikov Theorem [8, Theorem 76.1].

Theorem 5.10. *Suppose that $\omega_1 \leq \tau < \omega_2$ and that $(C_\alpha \mid \alpha < \tau)$ is a sequence of nontrivial countable (possibly finite) Σ-cyclic p-groups such that C_α is unbounded for each α such that $\alpha + 1 < \tau$. Then there exists a reduced abelian p-group A of cardinality ω_1 with $\tau(A) = \tau$ such that $A_\alpha \cong C_\alpha$ for all $\alpha < \tau$.*

Proof of Proposition 5.8. By counting nice $\mathbb{P}$-names, it follows that $\lambda_{\mathbb{P}}(\cong_p) \leq 2^{\omega_1}$. To see that $\lambda_{\mathbb{P}}(\cong_p) \geq 2^{\omega_1}$, for each sequence $\xi \in 2^{\omega_1}$, let $A(\xi)$ be a reduced abelian p-group of cardinality ω_1 with $\tau(A) = \omega_1$ such that for all $\alpha < \omega_1$,

$$A(\xi)_\alpha = \begin{cases} \bigoplus_{n \in \omega^+} C_{p^{2n}} & \text{if } \xi(\alpha) = 0; \\ \bigoplus_{n \in \omega^+} C_{p^{2n+1}} & \text{if } \xi(\alpha) = 1. \end{cases}$$

(The existence of such groups follows from Theorem 5.10.) Then we can suppose that each $A(\xi)$ has the form $\langle\, \omega_1, +_\xi \,\rangle$ for some group operation $+_\xi$ on the set ω_1. Let σ_ξ be a $\mathbb{P}$-name such that if $G \subseteq \mathbb{P}$ is a generic filter and $g = \bigcup G$, then $\sigma_\xi[G] = \langle\, \omega, \oplus_\xi \,\rangle \in \mathcal{A}_p^{V^{\mathbb{P}}}$, where

$$a \oplus_\xi b = c \qquad \Longleftrightarrow \qquad g(a) +_\xi g(b) = g(c).$$

Applying Theorem 4.1, we see that each σ_ξ is $\cong_p$-pinned; and also that if $\xi \neq \xi'$, then σ_ξ, $\sigma_{\xi'}$ are $\cong_p$-inequivalent. □

Proposition 5.9 is an easy consequence of the following result.

Proposition 5.11. $\lambda_{\mathbb{P}}(\equiv_p) = \omega_2$.

Proof. Let $\langle R_{p,m} \mid m \in \omega \rangle$ be our fixed sequence of representatives of the countably many bi-embeddability classes of nontrivial countable (possibly finite) Σ-cyclic p-groups, chosen so that $R_{p,0} = \bigoplus_{n\geq 1} C_{p^n}^{(\omega)}$ is the representative of the class of unbounded groups. Let I be the collection of all triples (α, m, d) with $\alpha < \omega_2$ and m, $d \in \omega$ such that:

- if $\alpha = 0$, then $m = d = 0$;
- if α is a limit ordinal, then $m = 0$.

For each $(\alpha, m, d) \in I$, let $A(\alpha, m, d)$ be an abelian p-group satisfying the following properties.

- $A(0,0,0) = \mathbb{Z}(p^\infty)^{(\omega)}$ is the divisible abelian p-group of rank ω.
- If α is a limit ordinal, then $A(\alpha, 0, 0)$ is a reduced abelian p-group of cardinality $|\alpha|$ such that for each $\gamma < \alpha$, the Ulm factor $A(\alpha, 0, 0)_\gamma$ is isomorphic to $R_{p,0}$.
- If $\alpha = \beta + 1$ is a successor ordinal, then $A(\alpha, m, 0)$ is a reduced abelian p-group of cardinality $|\alpha| + \omega$ such that for each $\gamma < \beta$, the Ulm factor $A(\alpha, n, 0)_\gamma$ is isomorphic to $R_{p,0}$ and such that the final Ulm factor $A(\alpha, n, 0)_\beta$ is isomorphic to $R_{p,m}$.
- If α, $m \neq 0$, then $A(\alpha, m, d) \cong A(\alpha, m, 0) \oplus \mathbb{Z}(p^\infty)^{(d)}$.

(The existence of such groups follows from Theorems 4.2 and 5.10.) In addition, we choose $A(\alpha, m, d)$ so that:

- if $\alpha < \omega_1$, then $A(\alpha, m, d) \in \mathcal{A}_p$;
- if $\omega_1 \leq \alpha < \omega_2$, then $A(\alpha, m, d)$ has the form $\langle \omega_1, +_{(\alpha,m,d)} \rangle$ for some group operation $+_{(\alpha,m,d)}$ on the set ω_1.

If $\alpha < \omega_1$, let $\sigma_{(\alpha,m,d)}$ be the canonical $\mathbb{P}$-name $\check{A}(\alpha, m, d)$ of $A(\alpha, m, d) \in \mathcal{A}_p$; and if $\omega_1 \leq \alpha < \omega_2$, let $\sigma_{(\alpha,m,d)}$ be the $\mathbb{P}$-name such that if $G \subseteq \mathbb{P}$ is a generic filter and $g = \bigcup G$, then $\sigma_{(\alpha,m,d)}[G] = \langle \omega, \oplus_{(\alpha,m,d)} \rangle \in \mathcal{A}_p^{V^{\mathbb{P}}}$, where

$$a \oplus_{(\alpha,m,d)} b = c \qquad \Longleftrightarrow \qquad g(a) +_{(\alpha,m,d)} g(b) = g(c).$$

Applying Theorem 4.1, we see that each $\sigma_{(\alpha,m,d)}$ is $\cong_p$-pinned and hence is also $\equiv_p$-pinned. Applying Theorem 4.11, we also see that if $(\alpha, m, d) \neq (\alpha', m', d')$, then $\sigma_{(\alpha,m,d)}$, $\sigma_{(\alpha',m',d')}$ are $\equiv_p$-inequivalent. Finally, by a second application of Theorem 4.11, since $\omega_1^{V^{\mathbb{P}}} = \omega_2^V$, it follows that if $G \subseteq \mathbb{P}$ is a generic filter and

$A \in \mathcal{A}_p^{V^{\mathbb{P}}}$, then there exists $(\alpha, m, d) \in I$ such that $\sigma_{(\alpha,m,d)}[G] \equiv_p A$; and this implies that if σ is any $\equiv_p$-pinned $\mathbb{P}$-name, then there exists $(\alpha, m, d) \in I$ such that $\sigma_{(\alpha,m,d)} \equiv_p \sigma$. Thus $\lambda_{\mathbb{P}}(\equiv_p) = \omega_2$. □

Proof of Proposition 5.9. Suppose that σ is an $\equiv_{TA}$-pinned $\mathbb{P}$-name; and for each prime p, let σ_p be a $\mathbb{P}$-name such that whenever $G \subseteq \mathbb{P}$ is a generic filter, then $\sigma_p[G]$ is the p-primary component of $\sigma[G]$. Then each σ_p is an $\equiv_p$-pinned $\mathbb{P}$-name. Furthermore, if σ' is a second $\equiv_{TA}$-pinned $\mathbb{P}$-name and σ'_p is the corresponding $\equiv_p$-pinned $\mathbb{P}$-name for each prime p, then

$$\sigma \equiv_{TA} \sigma' \qquad \Longleftrightarrow \qquad \sigma_p \equiv_p \sigma'_p \text{ for every prime } p.$$

Thus the result follows from Proposition 5.11. □

Appendix A

In this appendix, we will explain how to derive Theorem 4.11 from Corollary 5.4 of Barwise-Eklof [1]. First we need to define some invariants which play an important role in the work of Barwise-Eklof [1].

Suppose that A is a (not necessarily countable) abelian group. Then a set X of non-zero elements of A is said to be *independent* if whenever $x_1, \dots, x_k$ are distinct elements of X and $n_1, \dots, n_k$ are integers such that $n_1x_1 + \dots + n_kx_k = 0$, then then $n_ix_i = 0$ for all $1 \leq i \leq k$. By Fuchs [7, Theorem 16.3], if X, $Y \subseteq A$ are maximal independent sets, then $|X| = |Y|$; and so we can define the *rank* $\mathrm{rk}(A)$ of A to be the cardinality $|X|$ of any maximal independent subset $X \subseteq A$. Of course, this notation is consistent with our earlier use of the notation $\mathrm{rk}(A^{\tau(A)})$. Also, notice that if B is a subgroup of A and $X \subseteq B$ is a maximal independent subset of B, then X can be extended to a maximal independent subset X' of A. It follows that if $B \leqslant A$, then $\mathrm{rk}(B) \leq \mathrm{rk}(A)$.

Now suppose that A is a countable abelian p-group. Then for each countable ordinal α, we define the subgroup $p^\alpha A$ inductively by:

- $p^0 A = A$;
- $p^{\alpha+1} A = p(p^\alpha A)$;
- $p^\delta = \bigcap_{\alpha<\delta} p^\alpha A$, if δ is a limit ordinal.

Since A is countable, there exists a countable ordinal α such that $p^\alpha A = p^{\alpha+1} A$; and we define the *length* $\ell(A)$ to be the least such ordinal α. The relationship between the length $\ell(A)$ and the Ulm length $\tau(A)$ of a countable abelian p-group A is easily described. Let $\ell(A) = \omega\beta + n$, where $n \in \omega$. Then

$$\tau(A) = \begin{cases} \beta, & \text{if } n = 0; \\ \beta + 1, & \text{if } n > 0. \end{cases}$$

We are now ready to state the Barwise-Eklof [1, Corollary 5.4] characterization of the embeddability relation for countable abelian p-groups.

Theorem A.1. *If A and B are countable abelian p-groups, then A is embeddable into B if and only if* $\mathrm{rk}(p^\alpha A) \leq \mathrm{rk}(p^\alpha B)$ *for all countable ordinals $\alpha < \omega_1$.*

Of course, this implies the following characterization of the bi-embeddability relation for countable abelian p-groups.

Corollary A.2. *If A and B are countable abelian p-groups, then the following statements are equivalent:*

(i) *A and B are bi-embeddable.*
(ii) $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ *for all countable ordinals $\alpha < \omega_1$.*

Thus, in order to prove Theorem 4.11, it is enough to show that statement (A.2)(ii) is equivalent to the disjunction of statements (4.11)(a) and (4.11)(b). We will begin by considering the special case when both A and B are reduced countable abelian p-groups; i.e. when $A^{\tau(A)} = B^{\tau(B)} = 0$. Of course, in this special case, statement (4.11)(a) cannot hold.

First suppose that statement (A.2)(ii) holds; i.e. that $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ for all $\alpha < \omega_1$. Notice that if $\alpha < \beta < \omega_1$, then $p^\beta A \leqslant p^\alpha A$ and so $\mathrm{rk}(p^\beta A) \leq \mathrm{rk}(p^\alpha A)$. It follows that

$$\ell(A) = \text{ the least } \alpha < \omega_1 \text{ such that } \mathrm{rk}(p^\alpha A) = 0;$$

and hence there exists an ordinal $\ell < \omega_1$ such that $\ell(A) = \ell(B) = \ell$. Let $\ell = \omega\beta + n$, where $n \in \omega$.

Case 1: Suppose that $n = 0$ and that β is a limit ordinal. Then $\tau(A) = \tau(B) = \beta$ and statement (4.11)(b) holds.

Case 2: Suppose that $n = 0$ and that $\beta = \alpha + 1$ is a successor ordinal. Then it follows that:

- $\tau(A) = \tau(B) = \alpha + 1$;
- $p^{\omega\alpha} A = A^\alpha \cong A_\alpha$;
- $p^{\omega\alpha} B = B^\alpha \cong B_\alpha$.

In particular, since $p^{\omega\alpha} A$ is isomorphic to the Ulm factor A_α, it follows that $p^{\omega\alpha} A$ is a Σ-cyclic p-group. Furthermore, since $\mathrm{rk}(p^n(p^{\omega\alpha} A)) = \mathrm{rk}(p^{\omega\alpha+n} A) > 0$ for all $n \in \omega$, it follows that $p^{\omega\alpha} A$ is unbounded. Similary, we see that $p^{\omega\alpha} B$ is an unbounded Σ-cyclic p-group. Consequently, since the Ulm factors A_α and B_α are both countable unbounded Σ-cyclic p-groups, it follows that A_α and B_α are bi-embeddable. Thus statement (4.11)(b) holds.

Case 3: Suppose that $n > 0$. Then $\tau(A) = \tau(B) = \beta + 1$. Furthermore, arguing as in Case 2, we see that the Ulm factors A_β and B_β are both countable Σ-cyclic p-groups such that:

- $p^n A_\beta = p^n B_\beta = 0$;
- $\mathrm{rk}(p^m A_\beta) = \mathrm{rk}(p^m B_\beta) > 0$ for all $0 \leq m < n$.

It follows easily A_β and B_β are bi-embeddable. Thus statement (4.11)(b) holds.

Next suppose that statement (4.11)(b) holds. Thus A and B are reduced countable abelian p-groups such that:

- $\tau(A) = \tau(B)$;
- if $\tau(A) = \tau(B)$ is a successor ordinal $\beta + 1$, then the Ulm factors A_β and B_β are bi-embeddable.

In our analysis, we will make use of the following result of Barwise-Eklof [1, 2.6].

Lemma A.3. *Let G be a countable abelian p-group and suppose that $\ell(G) = \omega\gamma + n$, where $n \in \omega$. Then $\mathrm{rk}(p^\alpha G) = \omega$ for all $\alpha < \omega\gamma$.*

Case 1: Suppose that $\tau(A) = \tau(B)$ is a limit ordinal τ. Then $\ell(A) = \ell(B) = \omega\,\tau$. In particular, if $\omega\tau \leq \alpha < \omega_1$, then $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = 0$. Furthermore, applying Lemma A.3, we see that if $\alpha < \omega\,\tau$, then $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \omega$. Thus $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ for all $\alpha < \omega_1$.

Case 2: Suppose that $\tau(A) = \tau(B)$ is a successor ordinal $\beta + 1$ and that the Ulm factors A_β and B_β are bi-embeddable. Since A_β, B_β are Σ-cyclic and bi-embeddable, it follows that $\ell(A_\beta) = \ell(B_\beta) \leq \omega$ and that $\mathrm{rk}(p^m A_\beta) = \mathrm{rk}(p^m A_\beta)$ for all $0 \leq m < \omega$. (This special case of Corollary A.2 can easily be checked directly.) Note that $p^{\omega\beta} A = A^\beta \cong A_\beta$ and $p^{\omega\beta} B = B^\beta \cong B_\beta$. By Lemma A.3,

$$\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \omega$$

for all $0 \leq \alpha < \omega\beta$. Also for each $0 \leq m < \omega$,

$$\mathrm{rk}(p^{\omega\beta+m} A) = \mathrm{rk}(p^m A^\beta) = \mathrm{rk}(p^m B^\beta) = \mathrm{rk}(p^{\omega\beta+m} B).$$

Finally, $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = 0$ for all $\omega(\beta+1) \leq \alpha < \omega_1$. Thus $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ for all $\alpha < \omega_1$.

This completes the proof of Theorem 4.11 for the special case when both A and B are reduced countable abelian p-groups.

Now suppose that A and B are arbitrary (not necessarily reduced) countable abelian p-groups. Then we can express $A = A^{\tau(A)} \oplus C$ and $B = B^{\tau(B)} \oplus D$, where C, D are reduced abelian p-groups; and it is easily checked that:

- $\tau(A) = \tau(C)$;
- $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha C) + \mathrm{rk}(A^{\tau(A)})$ for all $\alpha < \omega_1$;
- the Ulm factors A_β and C_β are isomorphic for all $\beta < \tau(A) = \tau(C)$;

and the corresponding statements also hold for B, D.

First suppose that statement (A.2)(ii) holds; i.e. that $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ for all $\alpha < \omega_1$. Note that if $\alpha \geq \max\{\, \ell(A), \ell(B) \,\}$, then $p^\alpha A = A^{\tau(A)}$ and $p^\alpha B = B^{\tau(B)}$; and so

$$\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \mathrm{rk}(B^{\tau(B)}).$$

In particular, $\mathrm{rk}(A^{\tau(A)}) = \omega$ if and only $\mathrm{rk}(B^{\tau(B)}) = \omega$; and so we can suppose that there exists an integer $d \geq 0$ such that $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) = d$. Since

$$\mathrm{rk}(p^\alpha C) + d = \mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \mathrm{rk}(p^\alpha D) + d,$$

it follows that $\mathrm{rk}(p^\alpha C) = \mathrm{rk}(p^\alpha D)$ for all $\alpha < \omega_1$. Since C and D are reduced countable abelian p-groups, this implies that:

- $\tau(C) = \tau(D)$;

- if $\tau(C) = \tau(D)$ is a successor ordinal $\beta + 1$, then the Ulm factors C_β and D_β are bi-embeddable.

Since the Ulm factors A_γ and C_γ are isomorphic for all $\gamma < \tau(A) = \tau(C)$ and the Ulm factors B_γ and D_γ are isomorphic for all $\gamma < \tau(B) = \tau(D)$, it follows that statement (4.11)(b) holds.

Finally suppose that either statement (4.11)(a) holds or statement (4.11)(b) holds. Arguing as above, we see that if $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) = \omega$, then $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \omega$ for all $\alpha < \omega_1$. Hence we can suppose that there exists an integer $d \geq 0$ such that $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) = d$ and that:

- $\tau(A) = \tau(B)$;
- if $\tau(A) = \tau(B)$ is a successor ordinal $\beta + 1$, then the Ulm factors A_β and B_β are bi-embeddable.

Since $\tau(C) = \tau(A) = \tau(B) = \tau(D)$, the Ulm factors A_γ and C_γ are isomorphic for all $\gamma < \tau(A) = \tau(C)$, and the Ulm factors B_γ and D_γ are isomorphic for all $\gamma < \tau(B) = \tau(D)$, it follows that $\mathrm{rk}(p^\alpha C) = \mathrm{rk}(p^\alpha D)$ for all $\alpha < \omega_1$; and hence

$$\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha C) + d = \mathrm{rk}(p^\alpha D) + d = \mathrm{rk}(p^\alpha B)$$

for all $\alpha < \omega_1$. This completes the proof of Theorem 4.11.

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