POPA SUPERRIGIDITY AND COUNTABLE BOREL EQUIVALENCE RELATIONS

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Abstract. We present some applications of Popa's Superrigidity Theorem to the theory of countable Borel equivalence relations. In particular, we show that the universal countable Borel equivalence relation E_∞ is not essentially free.

1. Introduction

Let X be a standard Borel space; i.e. a Polish space equipped with its associated σ-algebra of Borel subsets. Then a Borel equivalence relation E on X is said to be *countable* iff every E-class is countable. For example, suppose that G is a countable group and that X is a standard Borel G-space; i.e. there exists a Borel action $(g, x) \mapsto g \cdot x$ of G on X. Then the corresponding G-orbit equivalence relation E^X_G is a countable Borel equivalence relation. Conversely, by a remarkable result of Feldman-Moore [7], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E^X_G$. However, it should be pointed out that the group G cannot be canonically recovered from E; and it is usually very difficult to determine whether two given Borel actions of a pair G, H of countable groups give rise to Borel bireducible orbit equivalence relations. Consequently, the fundamental question in the study of countable Borel equivalence relations concerns the extent to which the data (X, E^X_G) determines the group G and its action on X. In order for there to be any chance of recovering G from this data, it is necessary to assume the following extra hypotheses:

(i) G acts *freely* on X; i.e. $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$.

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(ii) There exists a G-invariant probability measure μ on X.

For example, by Dougherty-Jackson-Kechris [5], if (i) holds and (ii) fails, then for any countable group $H \supseteq G$, there exists a free Borel action of H on X such that $E_H^X = E_G^X$. On the other hand, by Miller [21], if E is an aperiodic countable Borel equivalence relation on the standard Borel space X, then there exist $2^{\aleph_0}$ nonisomorphic countable groups G such that $E = E_G^X$ for some everywhere faithful Borel action. Here a countable Borel equivalence relation E is said to be *aperiodic* iff every E-class is infinite; and a G-action is said to be *everywhere faithful* iff G acts faithfully on every orbit.

Of course, this raises the question of whether an arbitary countable Borel equivalence relation is Borel bireducible with an orbit equivalence E_G^X arising from an action which satisfies conditions (i) and (ii). It is easily seen that if E is any countable Borel equivalence relation on an uncountable standard Borel space, then there exists a countable group G and a standard Borel G-space X such that G preserves a nonatomic probability measure μ on X and $E \sim_B E_G^X$. But it remained unclear whether every countable Borel equivalence relation was essentially free.

Definition 1.1. Let E be a countable Borel equivalence relation on the standard Borel space X.

(i) E is said to be *free* iff there exists a countable group G with a free Borel action on X such that $E_G^X = E$.

(ii) E is said to be *essentially free* iff there exists a free countable Borel equivalence relation F such that $E \sim_B F$.

The collection of essentially free countable Borel equivalence relations satisfies the following closure properties.

Theorem 1.2 (Jackson-Kechris-Louveau [14])**.** *Suppose that E, F, E_n, $n \in \mathbb{N}$, are countable Borel equivalence relations.*

(a) *If $E \leq_B F$ and F is essentially free, then E is also essentially free.*

(b) *If $E \subseteq F$ and F is essentially free, then E is also essentially free.*

(c) *If E_n, $n \in \mathbb{N}$, are essentially free, then $\bigsqcup_{n\in\mathbb{N}} E_n$ is also essentially free.*

In particular, the question of whether every countable Borel equivalence relation is essentially free is equivalent to the question of whether the universal countable

Borel equivalence relation E_∞ is essentially free. This question is answered by the following result, which will be proved in Section 3. (As we shall see, Theorem 1.3 is an easy consequence of Popa's Cocyle Superrigidity Theorem [30].)

Theorem 1.3. *E_∞ is not essentially free.*

This paper is organized as follows. In Section 2, we shall recall some basic notions from the theory of countable Borel equivalence relations and ergodic theory. In Section 3, we shall state an easily applicable consequence of Popa's Cocycle Superrigidity Theorem which does not explicitly mention Borel cocyles. Using this result, we shall first prove that E_∞ is not essentially free; and then we shall give straightforward proofs that there exist both uncountably many free and also uncountably many non-essentially free countable Borel equivalence relations up to Borel bireducibility. Unfortunately the results of Section 3 do not provide any examples of "naturally occurring" non-essentially free countable Borel equivalence relations E such that $E \not\sim_B E_\infty$ and it remains an open problem to find an example of such an equivalence relation. In Section 4, we shall point out a potential source of such examples; namely, the weakly universal countable Borel equivalence relations. In Section 5, after a brief discussion of the notion of a Borel cocycle, we shall state Popa's Cocycle Superrigidity Theorem and then prove the easily applicable consequence of Section 3. Section 5 also includes a simple (modulo Popa's Theorem) proof of Adams' Theorem [1] that there exist countable Borel equivalence relations $E \subseteq F$ such that E, F are incomparable with respect to Borel reducibility. In Section 6, we shall prove that the isomorphism relation on the space of torsion-free abelian groups of finite rank is not countable universal. (Recall that in Thomas [36], it was shown that for each *fixed* $n \geq 1$, the isomorphism relation on the space of torsion-free abelian groups of rank n is not countable universal. However, the corresponding problem for the space of groups of finite rank remained open.) Finally, in Section 7, we shall study the orbit equivalence relations arising from the (not necessarily free) Borel actions of quasi-finite groups. In particular, we show that no such countable Borel equivalence relation is universal. The section also includes the proof of a technical group theoretic result that is needed in Section 3. We have also taken the opportunity throughout the paper to point out some of the many fundamental open problems which still remain in this area.

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2. Preliminaries

In this section, we shall recall some basic notions from the theory of countable Borel equivalence relations and ergodic theory.

2.1. **Borel equivalence relations.** For any unexplained notions or notation, see Jackson-Kechris-Louveau [14] or Hjorth-Kechris [12]. Here we shall only mention a few notions which some readers might be unfamilar with.

Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y respectively. Then a Borel map $f : X \to Y$ is said to be a *homomorphism* from E to F iff for all $x, y \in X$,

$$x \, E \, y \quad \text{implies} \quad f(x) \, F \, f(y).$$

If f satisfies the stronger property that for all $x, y \in X$,

$$x \, E \, y \quad \text{iff} \quad f(x) \, F \, f(y),$$

then f is said to be a *Borel reduction* and we write $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, then we write $E \sim_B F$ and say that E, F are Borel bireducible. In this case, there exists a *Borel bireduction* $f : X \to Y$ from E to F; i.e. a Borel reduction such that $\operatorname{ran} f \cap [y]_F \neq \emptyset$ for all $y \in Y$.

Let X, A be standard Borel spaces and suppose that $\{E_z \mid z \in A\}$ is a family of countable Borel equivalence relations on X such that the relation $R \subseteq X^2 \times A$, defined by

$$(x, x', z) \in R \quad \text{iff} \quad x \, E_z \, x',$$

is Borel. Then the corresponding *smooth disjoint union* is the countable Borel equivalence relation $E_A = \bigsqcup_{z \in A} E_z$ defined on $X \times A = \bigsqcup_{z \in A} X \times \{z\}$ by

$$(x, z) \, E_A \, (x', z') \quad \text{iff} \quad z = z' \text{ and } x \, E_z \, x'.$$

If $A = \{1, 2, \cdots, n\}$, then we often write $E_1 \oplus \cdots \oplus E_n$ instead of $E_1 \sqcup \cdots \sqcup E_n$.

In Sections 3 and 7, we shall refer to a Borel family $\{S_x \mid x \in 2^{\mathbb{N}}\}$ of finitely generated groups. By this, we mean the image of a Borel injection of the Cantor space $2^{\mathbb{N}}$ into the standard Borel space $\mathcal{G}$ of finitely generated groups. Of course, this means that $\{S_x \mid x \in 2^{\mathbb{N}}\}$ is a Borel subset of $\mathcal{G}$ and hence is also a standard Borel space.

2.2. **Ergodic theory.** Let G be a countably infinite group and let X be a standard Borel G-space. Throughout this paper, a probability measure on X will always mean a Borel probability measure; i.e. a measure which is defined on the collection of Borel subsets of X. The probability measure μ on X is *G-invariant* iff $\mu(g(A)) = \mu(A)$ for every $g \in G$ and Borel subset $A \subseteq X$. If μ is G-invariant, then the action of G on (X, μ) is said to be *ergodic* iff for every G-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. In this case, we shall also say that μ is an ergodic probability measure. The following characterization of ergodicity is well-known.

Theorem 2.1. *If μ is a G-invariant probability measure on the standard Borel G-space X, then the following statements are equivalent.*

(i) *The action of G on (X, μ) is ergodic.*
(ii) *If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel map, then there exists a Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant map.*

In this paper, we shall make use of two strong forms of ergodicity; namely, unique ergodicity and strong mixing.

The action of G on X is said to be *uniquely ergodic* iff there exists a unique G-invariant probability measure μ on X. In this case, it is well-known that μ must be ergodic. (For example, see Bekka-Mayer [4, Section I.3].)

If G is a countably infinite group and X is a standard Borel G-space with a G-invariant probability measure μ, then the action of G on (X, μ) is said to be *strongly mixing* iff for any two Borel subsets A, $B \subseteq X$, if $\langle g_n \mid n \geq 0 \rangle$ is a sequence of distinct elements of G, then

$$\lim_{n \to \infty} \mu(g_n(A) \cap B) = \mu(A)\mu(B).$$

A mixing action is necessarily ergodic. To see this, suppose that A is a G-invariant Borel subset of X. Then $g(A) \cap A = A$ for all $g \in G$. Hence if $\langle g_n \mid n \geq 0 \rangle$ is a sequence of distinct elements of G, then

$$\mu(A) = \lim_{n\to\infty} \mu(A) = \lim_{n\to\infty} \mu(g_n(A) \cap A) = \mu(A)\mu(A)$$

and so $\mu(A) = 0, 1$. Notice that if the action of G on (X, μ) is strongly mixing and H is an infinite subgroup of G, then the action of H is also strongly mixing and hence H acts ergodically on (X, μ).

Finally suppose that E is a countable Borel equivalence relation on the standard Borel space X and that μ is a probability measure on X. Then μ is said to be *E-invariant* iff μ is G-invariant for some (equivalently every) countable group G with a Borel action on X such that $E = E_G^X$.

3. Non-essentially free countable Borel equivalence relations

In this section, using Popa's Cocycle Superrigidity Theorem [30], we shall prove that the universal countable Borel equivalence relation E_∞ is not essentially free. Then we shall give a simple (modulo Popa's Theorem) proof of the Adams-Kechris Theorem [2] that there are uncountably many free countable Borel equivalence relations up to Borel bireducibility. Finally we shall prove that there are also uncountably many non-essentially free countable Borel equivalence relations up to Borel bireducibility. The actual statement of Popa's Cocycle Superrigidity Theorem will not be given until Section 5. In this section, we shall instead work with an easily applicable consequence of Popa's Cocycle Superrigidity Theorem which does not explicitly mention Borel cocycles. We shall begin by recalling some of the basic properties of the shift action.

Definition 3.1. Let G be a countably infinite group and consider the shift action on 2^G. Then the usual product probability measure μ on 2^G is G-invariant and the *free part* of the action

$$(2)^G = \{x \in 2^G \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G\}$$

has μ-measure 1. Let E_G be the corresponding orbit equivalence relation on $(2)^G$.

The following result is well-known. (For example, a proof can be found in either Hjorth-Kechris [12, Proposition A6.1] or Bekka-Mayer [4, Example I.2.8(ii)].)

Lemma 3.2. *The shift action of G on $((2)^G, \mu)$ is strongly mixing.*

In particular, it follows that G acts ergodically on $((2)^G, \mu)$.

Observation 3.3. *If G embeds into H, then $E_G \leq_B E_H$.*

Proof. To simplify notation, suppose that $G \leqslant H$. For each $f \in (2)^G$, define a corresponding element $f^* \in 2^H$ by

$$f^*(h) = \begin{cases} f(h), & \text{if } h \in G; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easily checked that $f^* \in (2)^H$ and that the map $f \mapsto f^*$ is a Borel reduction from E_G to E_H. □

Of course, the converse is not true in general. For example, $E_{\mathbb{Z}\oplus\mathbb{Z}} \leq_B E_{\mathbb{Z}}$, but $\mathbb{Z} \oplus \mathbb{Z}$ certainly does not embed into $\mathbb{Z}$. However, as we shall soon explain, Popa's Cocycle Superrigidity Theorem implies that the converse holds if suitable hypotheses are imposed upon the group G. But first we need to introduce two more definitions.

Definition 3.4. Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y and that μ is an E-invariant probability measure on X. Then:

(a) The Borel homomorphism $f : X \to Y$ from E to F is said to be *μ-trivial* iff there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single F-class. Otherwise, f is said to be *μ-nontrivial.*

(b) E is said to be *F-ergodic* iff every Borel homomorphism from E to F is μ-trivial.

Definition 3.5. If G, H are groups, then the group homomorphism $\pi : G \to H$ is a *virtual embedding* iff the kernel $\ker \pi$ is finite.

The following result, which we will prove in Section 5, is a simple consequence of Popa's Cocycle Superrigidity Theorem. The remainder of this section will consist of a number of easy applications to the theory of countable Borel equivalence relations.

Theorem 3.6. *Let $G = SL_3(\mathbb{Z}) \times S$, where S is* any *countable group. Suppose that H is any countable group and that Y is a free standard Borel H-space. If* *there exists a μ-nontrivial Borel homomorphism from E_G to E_H^Y, then there exists a virtual embedding $\pi : G \to H$.*

Remark 3.7. In particular, the conclusion of Theorem 3.6 holds if there exists a Borel subset $Z \subseteq (2)^G$ with $\mu(Z) = 1$ such that $E_G \restriction Z \leq_B E_H^Y$.

Corollary 3.8. *Suppose that S is a countable group with no nontrivial finite normal subgroups and let $G = SL_3(\mathbb{Z}) \times S$. If H is any countable group, then $E_G \leq_B E_H$ iff G embeds into H.*

Proof. In this case, G has no nontrivial finite normal subgroups and hence the result is an immediate consequence of Theorem 3.6. □

It is now easy to show that E_∞ is not essentially free.

Theorem 3.9. *If E is an essentially free countable Borel equivalence relation, then there exists a countable group G such that $E_G \nleq_B E$.*

Proof. Clearly we can suppose that $E = E_H^X$ is realised by a free Borel action of the countable group H on the standard Borel space X. By B.H. Neumann [23], there exist uncountably many finitely generated groups. Hence there exists a finitely generated group L which does not embed into H. Let S be the free product $L * \mathbb{Z}$ and let $G = SL_3(\mathbb{Z}) \times S$. Then S has no nontrivial finite normal subgroups and clearly G does not embed into H. Hence $E_G \nleq_B E_H^X$. □

Corollary 3.10. *The class of essentially free countable Borel equivalence relations does not admit a universal element. In particular, E_∞ is not essentially free.*

□

Next we shall give a simple (modulo Popa's Theorem) proof of the following result.

Theorem 3.11 (Adams-Kechris [2]). *There exist uncountably many free countable Borel equivalence relations up to Borel bireducibility.*

To see this, let $\mathbb{P}$ be the set of primes and for each prime $p \in \mathbb{P}$, let $A_p = \bigoplus_{i=0}^{\infty} C_p$ be the direct sum of countably many copies of the cyclic group C_p of order p. For each subset $C \subseteq \mathbb{P}$, let

$$G_C = SL_3(\mathbb{Z}) \times \bigoplus_{p \in C} A_p.$$

Then Theorem 3.11 is an immediate consequence of the following:

Lemma 3.12. *If C, $D \subseteq \mathbb{P}$, then $E_{G_C} \leq_B E_{G_D}$ iff $C \subseteq D$.*

Proof. If $C \subseteq D$, then $G_C \leqslant G_D$ and hence $E_{G_C} \leq_B E_{G_D}$. Conversely, applying Theorem 3.6, if $E_{G_C} \leq_B E_{G_D}$, then there exists a virtual embedding $\pi : G_C \to G_D$. It is well-known that $SL_3(\mathbb{Z})$ contains a torsion-free subgroup of finite index. (For example, see Wehrfritz [38, Corollary 4.8].) It follows easily that for each $p \in C$, the cyclic group C_p embeds into $\bigoplus_{q \in D} A_q$ and this implies that $p \in D$. □

In the remainder of this section, we shall present a proof of the following result.

Theorem 3.13. *There exist uncountably many non-essentially free countable Borel equivalence relations up to Borel bireducibility.*

We shall make use of the following group-theoretic result, which will be proved in Section 7.

Definition 3.14. The groups G, H are *isomorphic up to finite kernels* iff there exist finite normal subgroups $N \trianglelefteq G$, $M \trianglelefteq H$ such that $G/N \cong H/M$.

Proposition 3.15. *There exists a Borel family $\{S_x \mid x \in 2^{\mathbb{N}}\}$ of finitely generated groups such that if $G_x = SL_3(\mathbb{Z}) \times S_x$, then the following conditions hold:*

(i) *If $x \neq y$, then G_x and G_y are not isomorphic up to finite kernels.*
(ii) *If $x \neq y$, then G_x does not virtually embed into G_y.*

Definition 3.16. For each Borel subset $A \subseteq 2^{\mathbb{N}}$, let $E_A = \bigsqcup_{x \in A} E_{G_x}$ be the corresponding smooth disjoint union of the countable Borel equivalence relations $\{E_{G_x} \mid x \in A\}$.

Theorem 3.13 is an immediate consequence of the following two lemmas.

Lemma 3.17. *If the Borel subset $A \subseteq 2^{\mathbb{N}}$ is uncountable, then E_A is not essentially free.*

Proof. Suppose that $E_A \leq_B E_H^Y$, where H is a countable group and Y is a free standard Borel H-space. Then for each $x \in A$, we have that $E_{G_x} \leq_B E_H^Y$ and so there exists a virtual embedding $\pi_x : G_x \to H$. Since A is uncountable, there exist $x \neq y \in A$ such that $\pi_x[G_x] = \pi_y[G_y]$. But then G_x, G_y are isomorphic up to finite kernels, which is a contradiction. □

Lemma 3.18. *$E_A \leq_B E_B$ iff $A \subseteq B$.*

Proof. Clearly if $A \subseteq B$, then $E_A \leq_B E_B$. Conversely, suppose that $E_A \leq_B E_B$. For the sake of contradiction, suppose that $A \nsubseteq B$ and let $x \in A \smallsetminus B$. Then there exists a Borel reduction

$$f : (2)^{G_x} \to \bigsqcup_{y \in B} (2)^{G_y}$$

from E_{G_x} to E_B. Since G_x acts ergodically on $((2)^{G_x}, \mu_x)$, there exists a Borel subset $Z \subseteq (2)^{G_x}$ with $\mu_x(Z) = 1$ such that f maps Z to a *fixed* $(2)^{G_y}$. This yields a μ_x-nontrivial Borel homomorphism from E_{G_x} to E_{G_y} and hence G_x virtually embeds into G_y, which is a contradiction. □

Of course, it is a little disappointing that the equivalence relations in the above proof of Theorem 3.13 are all smooth disjoint unions of free relations. However, it may not be possible to avoid this, since it remains conceivable that *every* countable Borel equivalence relation is Borel bireducible with a smooth disjoint union of free countable Borel equivalence relations. Arguing as in Jackson-Kechris-Louveau [14], this question is easily seen to be equivalent to the following special case.

Question 3.19. Is E_∞ Borel bireducible with a smooth disjoint union of free countable Borel equivalence relations?

Question 3.19 is very closely related to the following well-known open problem.

Question 3.20. Suppose that $E = \bigsqcup_{z \in A} E_z$ is a smooth disjoint union of the countable Borel equivalence relations $\{E_z \mid z \in A\}$. If E is countable universal, does there necessarily exist an element $z \in A$ such that E_z is countable universal?

(It should be pointed out that Question 3.20 remains open even in the special case when $A = \{1, 2\}$. See Jackson-Kechris-Louveau [14].) To see the connection between Questions 3.19 and 3.20, suppose that $E = \bigsqcup_{z \in A} E_z$ is the smooth disjoint union of the free countable Borel equivalence relations $\{E_z \mid z \in A\}$ and that E is countable universal. Then by Corollary 3.10, none of the relations E_z is countable universal and hence E is a counterexample to Question 3.20. On the other hand, a positive answer to Question 3.20 would follow easily from the existence of a "strongly universal" countable Borel equivalence relation.

Definition 3.21. Suppose that E is a countable Borel equivalence relation on the standard Borel space X with invariant ergodic probability measure μ. Then E is *strongly universal* iff $E \restriction Y$ is universal for every Borel subset $Y \subseteq X$ with $\mu(Y) = 1$.

Question 3.22. Does there exist a strongly universal countable Borel equivalence relation? Or does the complexity of a universal countable Borel equivalence relation always concentrate on a measure 0 subset?

Of course, a positive answer to the following question would rule out the existence of a strongly universal countable Borel equivalence relation.

Question 3.23. Suppose that E is a countable Borel equivalence relation on the standard Borel space X with invariant ergodic probability measure μ. Does there always exist a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is essentially free?

4. Weak Borel Reducibility

Unfortunately the results of the previous section do not provide any examples of "naturally occurring" non-essentially free countable Borel equivalence relations E such that $E \not\sim_B E_\infty$ and it remains an open problem to find an example of such an equivalence relation. In this section, we shall point out a potential source of such examples; namely, the weakly universal countable Borel equivalence relations. The material in this section is due to Alexander Kechris, with the exception of the crucial Proposition 4.10 which is due to Ben Miller.

Definition 4.1. Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y respectively. Then E is *weakly Borel reducible* to F, written $E \leq_B^w F$, iff there exists a countable-to-one Borel homomorphism

$f : X \to Y$ from E to F. In this case, we say that f is a *weak Borel reduction* from E to F.

Remark 4.2. If $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X, then the identity map $id : X \to X$ is a weak Borel reduction from E to F. By Adams [1], there exists a pair $E \subseteq F$ of countable Borel equivalence relations such that E, F are incomparable with respect to Borel reducibility. In particular, these provide examples of countable Borel equivalence relations E, F such that $E \leq_B^w F$ but $E \nleq_B F$. We shall present a simple (modulo Popa's Theorem) proof of Adams' Theorem in Section 5.

Remark 4.3. Suppose that μ is a nonatomic E-invariant probability measure on X. Clearly if E is F-ergodic, then $E \nleq_B^w F$. It is easily seen that the converse does not hold. For example, suppose that ν is an F-invariant probability measure on Y and regard ν as an $(E \sqcup F)$-invariant probability measure on $X \sqcup Y$. With this measure, $E \sqcup F$ is certainly not F-ergodic. However, if E is F-ergodic, then $(E \sqcup F) \nleq_B^w F$. Of course, this example is a little unsatisfactory since the obstruction to weak reducibility is once again an instance of the stronger notion of F-ergodicity. In Appendix A, we shall present an more satisfactory example consisting of a pair E, F of countable Borel equivalence relations such that:

- there exists a unique E-invariant probability measure on X,
- $E \nleq_B^w F$ but E is not F-ergodic.

The following elegant characterization of weak Borel reducibility will be proved in the second half of this section.

Theorem 4.4. *If E, F are countable Borel equivalence relations on the uncountable standard Borel spaces X, Y respectively, then the following are equivalent:*

(a) $E \leq_B^w F$.
(b) *There exists a countable Borel equivalence relation $R \supseteq E$ on X such that $R \leq_B F$.*
(c) *There exists a countable Borel equivalence relation $S \subseteq F$ on Y such that $E \leq_B S$.*

Definition 4.5. A countable Borel equivalence relation E is said to be *weakly universal* iff $F \leq_B^w E$ for every countable Borel equivalence relation F.

Of course, it also makes sense to consider weakly treeable relations, weakly hyperfinite relations, etc. However, applying Theorem 4.4 and Jackson-Kechris-Louveau [14, Proposition 3.3], it follows that every weakly treeable countable Borel equivalence relation is treeable. Similarly, every weakly hyperfinite countable Borel equivalence relation is hyperfinite. On the other hand, by Theorem 4.7, the corresponding problem for weakly universal relations is equivalent to a well-known open problem of Hjorth [3].

Question 4.6 (Hjorth)**.** Does there exist a weakly universal countable Borel equivalence relation E which is *not* countable universal?

Theorem 4.7. *Suppose that E is a countable Borel equivalence relation on the standard Borel space X. Then E is weakly universal iff there exists a universal countable Borel equivalence relation $R \subseteq E$.*

Proof. Clearly E is weakly universal iff $E_\infty \leq_B^w E$. Hence the result follows from Theorem 4.4. □

Corollary 4.8. *If E is a weakly universal countable Borel equivalence relation, then E is not essentially free.*

Proof. This is an immediate consequence of Theorems 4.7 and 1.2(b), together with Corollary 3.10. □

Corollary 4.9. *The Turing equivalence relation $\equiv_T$ on $2^{\mathbb{N}}$ is weakly universal and hence is not essentially free.*

Proof. Let $\mathbb{F}_2$ be the free group on two generators. Then E_∞ is the orbit equivalence relation arising from the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. (Of course, it is essential here to consider the action on the entire space $2^{\mathbb{F}_2}$ rather than on just the free part $(2)^{\mathbb{F}_2}$ of the action.) Identifying $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$ and hence $\equiv_T$ is weakly universal. □

There are currently no known techniques for distinguishing between "naturally occurring" non-essentially free countable Borel equivalence relations up to Borel bireducibility. In particular, it is unknown whether the Turing equivalence relation $\equiv_T$ is countable universal. By Dougherty-Kechris [6], if this is indeed the case, then the well-known Martin Conjecture (also known as the 5th Victoria Delfino Problem [16]) fails. This suggests that the Martin Conjecture is either extremely difficult or else is false.

The remainder of this section is devoted to the proof of Theorem 4.4. Suppose that E, F are countable Borel equivalence relations on the uncountable standard Borel spaces X, Y respectively.

$(a) \Rightarrow (b)$. Suppose that $f : X \to Y$ is a weak Borel reduction from E to F and let $R = f^{-1}(F)$. Then $R \supseteq E$ is a countable Borel equivalence relation and f is a Borel reduction from R to F.

$(c) \Rightarrow (a)$. If $f : X \to Y$ is a Borel reduction from E to S, then f is a weak reduction from E to F.

$(b) \Rightarrow (c)$. This is an immediate consequence of the following somewhat surprising result. (At least, my first instinct was to seek a counterexample.)

Proposition 4.10 (Miller). *Suppose that E, F are countable Borel equivalence relations on the uncountable standard Borel spaces X, Y respectively and that $E \leq_B F$. Then for every Borel equivalence relation $E' \subseteq E$, there exists a Borel equivalence relation $F' \subseteq F$ such that $E' \sim_B F'$.*

Recall that if F is a countable Borel equivalence relation on the standard Borel space Y, then $F \times I_\infty$ denotes the countable Borel equivalence on $Y \times \mathbb{N}$ defined by

$$(y, n)\,(F \times I_\infty)\,(z, m) \quad \text{iff} \quad y\, F\, z.$$

Of course, the Borel map $(y, n) \mapsto y$ witnesses that $(F \times I_\infty) \sim_B F$.

Lemma 4.11. *Suppose that E, F are countable Borel equivalence relations on the uncountable standard Borel spaces X, Y respectively and that $E \leq_B F$. Then for every Borel equivalence relation $E' \subseteq E$, there exists a Borel equivalence relation $R \subseteq (F \times I_\infty)$ such that $E' \sim_B R$.*

Proof. Let $f : X \to Y$ be a Borel reduction from E to F. By the Lusin-Novikov uniformization theorem [15, Theorem 18.10], there exists a partition $\bigsqcup_{n \in \mathbb{N}} X_n$ of X into Borel subsets such that each $f \restriction X_n$ is injective. Hence we can define an injective Borel reduction $\varphi : X \to Y \times \mathbb{N}$ from E to $F \times I_\infty$ by

$$\varphi(x) = (f(x), n), \quad \text{where } x \in X_n.$$

Let $E' \subseteq E$ be any Borel equivalence relation and let R' be the corresponding Borel equivalence relation on $\varphi(X)$ defined by

$$\varphi(x)\, R'\, \varphi(w) \quad \text{iff} \quad x\, E'\, w.$$

Then we can extend R' to a Borel equivalence relation R on $Y \times \mathbb{N}$ by letting $R \restriction ((Y \times \mathbb{N}) \smallsetminus \varphi(X))$ be the identity relation. It is clear that $R \subseteq (F \times I_\infty)$ and that $E' \sim_B R'$. Since X is uncountable, it follows that $E' \sim_B (E' \sqcup id_{(Y\times\mathbb{N})\smallsetminus\varphi(X)})$ and hence $E' \sim_B R$. □

Thus Proposition 4.10 follows from the following special case.

Proposition 4.12. *Suppose that F is a countable Borel equivalence relation on the uncountable standard Borel space Y. Then for every Borel equivalence relation $R \subseteq (F \times I_\infty)$, there exists a Borel equivalence relation $S \subseteq F$ such that $R \sim_B S$.*

Clearly we can suppose that both F and R are nonsmooth. Let A be the Borel subset of those $y \in Y$ such that $[y]_F$ is finite. Then $F \restriction A$ is smooth and there is an injective Borel reduction $Y \to Y \smallsetminus A$ from F to $F \restriction (Y \smallsetminus A)$. Hence, arguing as in Lemma 4.11, we can suppose that F is aperiodic. From now on, let $<$ be a fixed Borel linear ordering of Y and let $\prec$ be the Borel ordering on $Y \times \mathbb{N}$ defined by

$$(y, n) \prec (z, m) \quad \text{iff} \quad n < m \text{ or } (\ n = m \text{ and } y < z\ ).$$

Lemma 4.13. *There exists a partition $\bigsqcup_{n\in\mathbb{N}} B_n$ of Y into Borel subsets such that each B_n has measure exactly $1/2^{n+1}$ with respect to* every *F-invariant probability measure on Y.*

Proof. By Kechris-Miller [16, Proposition 7.4], since F is aperiodic, there exists a Borel equivalence relation $F_0 \subset F$, all of whose classes have cardinality exactly 2. Let i_0 be the fixed-point free Borel involution which interchanges the elements of each F_0-class and let B_0 be the Borel subset of Y consisting of the $<$-least element of each i_0-orbit. Note that $F \restriction i_0(B_0)$ is also aperiodic. Hence there also exists a fixed-point free involution i_1 on $i_0(B_0)$ with $\mathrm{graph}(i_1) \subset F$ and we can let B_1 consist of the $<$-least element of each i_1-orbit. Continuing in this fashion, the result follows. □

Let $R \subseteq (F \times I_\infty)$ be a Borel equivalence relation and let $\pi : Y \times \mathbb{N} \to Y$ be the canonical Borel bireduction from $F \times I_\infty$ to F defined by $(y, n) \mapsto y$. For each $n \in \mathbb{N}$, let $Y_n = Y \times \{n\}$. Define $Z \subseteq Y \times \mathbb{N}$ by

$$z \in Z \quad \text{iff} \quad \text{for all } n \in \mathbb{N},\ [z]_R \cap Y_n \text{ is either empty or infinite.}$$

Clearly $R \restriction ((Y \times \mathbb{N}) \smallsetminus Z)$ is smooth. For each $n \in \mathbb{N}$, let $Z_n = Z \cap Y_n$. Then $R \restriction Z_n$ is an aperiodic countable Borel equivalence. Hence, appealing once again to Kechris-Miller [16, Proposition 7.4], there exists a Borel equivalence relation $R_n \subseteq R \restriction Z_n$, all of whose classes are of cardinality exactly 2^{n+1}. Let $\varphi_n : Z_n \to Z_n$ be the Borel map which sends each $z \in Z_n$ to the $\prec$-least element of $[z]_{R_n}$. Note that $\pi(\varphi_n(Z_n))$ has measure at most $1/2^{n+1}$ with respect to *every* F-invariant probability measure on Y.

Examining the proof of Kechris-Miller [16, Lemma 7.10], we see that the following result holds. (Recall that if $C \subseteq X$ is a Borel subset, then $E \restriction C$ is compressible iff $\mu(C) = 0$ for every ergodic E-invariant probability measure μ on X.)

Lemma 4.14. *Let E be a countable Borel equivalence relation on the standard Borel space X and suppose that A, $B \subseteq X$ are Borel subsets such that $\mu(A) \leq \mu(B)$ for* every *E-invariant probability measure μ on X. Then there exists:*

- *an E-invariant Borel subset $C \subseteq X$, and*
- *a Borel injection $\psi : A \smallsetminus C \to B \smallsetminus C$*

such that $E \restriction C$ is compressible and $\operatorname{graph}(\psi) \subset E$*.*

Thus there exists an F-invariant Borel subset $C \subseteq Y$ and Borel injections

$$\psi_n : (\pi(\varphi_n(Z_n)) \smallsetminus C) \to (B_n \smallsetminus C)$$

such that $F \restriction C$ is compressible and each $\operatorname{graph}(\psi_n) \subset F$. By Dougherty-Jackson-Kechris [5, Proposition 2.5], since $F \restriction C$ is compressible, $F \restriction C \cong (F \restriction C) \times I_\infty$ and hence there exist Borel injections $\psi'_n : C \to C$ with $\operatorname{graph}(\psi'_n) \subset F$ such that $\psi'_n(C) \cap \psi'_m(C) = \emptyset$ for all $n \neq m$. Consider the Borel map $\theta : Z \to Y$ defined by

$$\theta(z) = \begin{cases} \psi_n \circ \pi \circ \varphi_n(z) & \text{if } z \in Z_n \text{ and } \pi \circ \varphi_n(z) \notin C; \\ \psi'_n \circ \pi \circ \varphi_n(z) & \text{if } z \in Z_n \text{ and } \pi \circ \varphi_n(z) \in C. \end{cases}$$

Notice that if $z, z' \in Z$ and $\theta(z) = \theta(z')$, then $z, z' \in Z_n$ for some $n \in \mathbb{N}$ and $z\, R\, z'$. Hence we can define an equivalence relation $S' \subseteq F$ on $\theta(Z)$ by

$$\theta(z)\, S'\, \theta(w) \quad \text{iff} \quad z\, R\, w.$$

Clearly θ is a Borel bireduction from $R \restriction Z$ to S' and so $R \restriction Z \sim_B S'$. Extend S' to a Borel equivalence relation S on Y by letting $S \restriction (Y \smallsetminus \theta(Z))$ be the identity relation. Since $R \restriction ((Y \times \mathbb{N}) \smallsetminus Z)$ is smooth and R is nonsmooth, it follows that

$$R \sim_B R \restriction Z \sim_B S' \sim_B S.$$

This completes the proof of Proposition 4.12.

5. Popa's cocycle superrigidity theorem

In this section, after a brief discussion of the notion of a Borel cocycle, we shall state Popa's Cocycle Superrigidity Theorem and then present the proof of Theorem 3.6. This section also includes a simple (modulo Popa's Theorem) proof of Adams' Theorem [1] that there exist countable Borel equivalence relations $E \subseteq F$ such that E, F are incomparable with respect to Borel reducibility.

We shall begin by recalling the notion of a Borel cocycle. Suppose that G is a countable group and that X is a standard Borel G-space with an invariant probability measure μ.

Definition 5.1. If H is a countable group, then a Borel function $\alpha : G \times X \to H$ is called a *cocycle* if for all $g, h \in G$,

$$\alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Cocycles typically (but not always) arise in the following manner. Suppose that Y is a standard Borel H-space on which H acts freely and that $f : X \to Y$ is a Borel homomorphism between the corresponding orbit equivalence relations E_G^X and E_H^Y. Then we can define a Borel cocycle $\alpha : G \times X \to H$ by setting

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

Notice that if $\alpha(g, x) = \alpha(g)$ only depends on the g-variable, then $\alpha : G \to H$ is a group homomorphism and $(G, X) \xrightarrow{\alpha, f} (H, Y)$ is a permutation group homomorphism. Even if a given Borel cocycle is not a group homomorphism, there remains the possibility that the cocycle corresponding to a "suitably adjusted" perturbation f' might be. More precisely, suppose that $b : X \to H$ is a Borel function and that $f' : X \to Y$ is defined by $f'(x) = b(x) \cdot f(x)$. Then f' is also a Borel homomorphism between E_G^X and E_H^Y; and the corresponding cocycle $\beta : G \times X \to H$ satisfies

$$\beta(g, x) = b(g \cdot x)\alpha(g, x)b(x)^{-1}$$

for all $g \in G$ and $x \in X$. This motivates the following definition.

Definition 5.2. The cocycles α, $\beta : G \times X \to H$ are *equivalent* iff there exist a Borel function $b : X \to H$ such that for all $g \in G$,

$$\beta(g, x) = b(g \cdot x)\alpha(g, x)b(x)^{-1} \quad \text{for } \mu\text{-a.e. } x \in X.$$

Of course, it is not true that an arbitrary Borel cocycle is equivalent to a group homomorphism. However, the following Popa Cocyle Superrigidity Theorem says that with suitable hypotheses on the group G and the G-space (X, μ), every Borel cocycle from $G \times X$ into an *arbitrary* countable group is equivalent to a group homomorphism. (In contrast, the earlier superrigidity theorems of Zimmer [39], Furman [9], Monod-Shalom [22] and Hjorth-Kechris [12] imposed strong restrictions on either the target group H or else on the cocycle α itself. For example, while Furman imposes no hypotheses on H, he does require that the relevant cocycle arise from a weak orbit equivalence.)

Theorem 5.3 (Popa [30])**.** *Let Γ be a countably infinite Kazhdan group and let G, $\mathbb{G}$ be countable groups such that $\Gamma \trianglelefteq G \leqslant \mathbb{G}$. If H is any countable group, then every Borel cocycle*

$$\alpha : G \times (2)^{\mathbb{G}} \to H$$

is equivalent to a group homomorphism of G into H.

Remark 5.4. In most applications, we can let $\mathbb{G} = G$. In this paper, we shall always take the Kazhdan group Γ to be either $SL_3(\mathbb{Z})$ or else a subgroup of finite index in $SL_3(\mathbb{Z})$. (For a proof that these groups do indeed satisfy the Kazhdan property, see Lubotzky [19, Chapter 3] or Zimmer [39, Chapter 7].)

Remark 5.5. It should be pointed out that this is not the most general statement of Popa's Cocyle Superrigidity Theorem. (For example, in his later paper [31], Popa has extended his result to cover nonamenable groups with infinite centers. Of course, this includes every group of the form $\mathbb{Z} \times G$, where G is an arbitrary countable nonamenable group.) However, there are currently no applications to the theory of countable Borel equivalence relations which do not follow from Theorem 5.3.

Remark 5.6. Popa's original proof [30] of Theorem 5.3 was discovered within the framework of Operator Algebra theory. More recently, Furman [9] has given a self-contained purely ergodic-theoretic presentation of Popa's proof.

Proof of Theorem 3.6. Let $G = SL_3(\mathbb{Z}) \times S$, where S is any countable group. Let H be a countable group and let Y be a free standard Borel H-space. Suppose that $f : (2)^G \to Y$ is a μ-nontrivial Borel homomorphism from E_G to E_H^Y. Then we can define a Borel cocycle $\alpha : G \times (2)^G \to H$ by

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

By Theorem 5.3, after deleting a nullset and slightly adjusting f if necessary, we can suppose that $\alpha : G \to H$ is a group homomorphism. Suppose that $N = \ker \alpha$ is infinite. Since the action of G is strongly mixing, it follows that N acts ergodically on $((2)^G, \mu)$. But this means that the N-invariant function $f : (2)^G \to Y$ must be μ-a.e. constant, which is a contradiction. Hence $N = \ker \alpha$ is finite and $\alpha : G \to H$ is a virtual embedding. □

In the remainder of this section, we shall use Popa's Superrigidity Theorem to give a simple proof of the following theorem.

Theorem 5.7 (Adams [1])**.** *There exists a pair of countable Borel equivalence relations $E \subseteq F$ on a standard Borel space X such that E, F are incomparable with respect to Borel reducibility.*

Adams' original proof [1] was based on an elegant application of the unique ergodicity of actions arising from dense embeddings of countable groups into compact groups. In [12], Hjorth-Kechris used the following result to apply Adams' idea to the strongly mixing actions of a suitably chosen pair of groups $T \leqslant S$ on $(2)^S$. As we shall see next, Popa's Superrigidity Theorem allows us to choose a particularly simple pair of such groups.

Lemma 5.8 (Hjorth-Kechris [12])**.** *Suppose that G is a countable group and that (X, μ) is a standard Borel G-space with invariant probability measure μ. If the action of G on (X, μ) is strongly mixing, then there exists a G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that the action of* every *infinite finitely generated subgroup of G on (X_0, μ) is uniquely ergodic.*

From now on, let $S = SL_3(\mathbb{Z})$ and let T be a proper subgroup of finite index. For example, we could let T be the kernel of the homomorphism $\varphi : SL_3(\mathbb{Z}) \to SL_3(\mathbb{F}_7)$. Then S and T are both Kazhdan groups and hence we can apply Theorem 5.3 to the strongly mixing actions of S and T on $((2)^S, \mu)$. Let $X \subseteq (2)^S$ be an S-invariant Borel subset with $\mu(X) = 1$ such that the action of *every* infinite finitely generated subgroup of S on (X, μ) is uniquely ergodic. Let $E \subseteq F$ be the the orbit equivalence relations corresponding to the free actions of T, S respectively on (X, μ). We shall prove that E, F are are incomparable with respect to Borel reducibility.

To see this, first suppose that $F \leq_B E$. Then Theorem 5.3 implies that there exists a virtual embedding $\pi : S = SL_3(\mathbb{Z}) \to T$. But the following lemma shows that this is impossible. (Lemma 5.9 is a consequence of an elementary rigidity result of Steinberg [33, Theorem 6]. In particular, it does not require an application of the deep results of Margulis [20].)

Lemma 5.9. *Suppose that G is a (not necessarily proper) subgroup of finite index in $SL_3(\mathbb{Z})$. Then:*

(a) *G has no nontrivial finite normal subgroups.*
(b) *G does not embed into any of its proper subgroups of finite index.*

Next suppose that $E \leq_B F$ and let $f : X \to X$ be a Borel reduction from E to F. Then we can define a corresponding Borel cocycle $\alpha : T \times X \to S$ by

$$\alpha(t, x) = \text{ the unique } s \in S \text{ such that } s \cdot f(x) = f(t \cdot x).$$

Applying Theorem 5.3, after deleting a nullset and slightly adjusting f if necessary, we can suppose that $\alpha : T \to S$ is a virtual embedding. Since T has no finite normal subgroups, it follows that α is an embedding; and since $S \ncong T$, it follows that $\alpha(T)$ is a proper subgroup of S. Because the actions of S, T on (X, μ) are free and

$$\alpha(t) \cdot f(x) = f(t \cdot x) \quad \text{ for } t \in T, x \in X,$$

it also follows that f is an injection. Thus we have an embedding $(T, X) \xrightarrow{\alpha, f} (S, X)$ of permutation groups and so we can define an $\alpha(T)$-invariant probability measure $\nu = f_* \mu$ on X by $\nu(A) = \mu(f^{-1}(A))$. Since the action of $\alpha(T)$ on (X, μ) is uniquely ergodic, we must have that $\nu = \mu$ and hence $\mu(f(X)) = 1$. Of course, this means that

$$\mu(\, f(X) \cap s \cdot f(X)\,) = 1$$

for all $s \in S$. In particular, choosing $s \in S \smallsetminus \alpha(T)$, there exist $x, y \in X$ such that

$$f(x) = s \cdot f(y) \in f(X) \cap s \cdot f(X).$$

Then $f(x)\; F\; f(y)$ and so $x \; E \; y$. Hence there exists $t \in T$ such that $x = t \cdot y$. It follows that

$$\alpha(t) \cdot f(y) = f(t \cdot y) = f(x) = s \cdot f(y)$$

and so $s^{-1}\alpha(t) \cdot f(y) = f(y)$, which contradicts the fact that S acts freely on X. Hence $E \not\leq_B F$. This completes the proof of Theorem 5.7.

Remark 5.10. A minor variant of the above argument shows that if $f : X \to X$ is a Borel reduction from F to F, then $\mu(f(X)) > 0$ and hence $\mu(S \cdot f(X)) = 1$. It follows easily that

$$F <_B F \oplus F <_B F \oplus F \oplus F <_B \cdots$$

For more details, see Thomas [34] or Hjorth-Kechris [12, Theorem 3.9].

6. Torsion-free abelian groups of finite rank

In this section, we shall consider the complexity of the isomorphism relation on the standard Borel space of torsion-free abelian groups of finite rank. Recall that, up to isomorphism, the torsion-free abelian groups A of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbb{Q}^n$ which contain n linearly independent elements. Thus the classification problem for the torsion-free abelian groups of rank n can be naturally identified with the corresponding problem for the standard Borel space

$$R(\mathbb{Q}^n) = \{A \leqslant \mathbb{Q}^n \mid A \text{ contains } n \text{ linearly independent elements}\}.$$

Letting $\cong_n$ denote the isomorphism relation on $R(\mathbb{Q}^n)$, it is easily checked that if A, $B \in R(\mathbb{Q}^n)$, then

$$A \cong_n B \quad \text{iff} \quad \text{there exists } \varphi \in GL_n(\mathbb{Q}) \text{ such that } \varphi(A) = B.$$

Thus $\cong_n$ is a countable Borel equivalence relation. In Thomas [36], making essential use of the Zimmer Superrigidity Theorem [39], together with the ideas of Adams-Kechris [2] and Hjorth [11], it was shown that

$$(\cong_1) <_B (\cong_2) <_B \cdots <_B (\cong_n) <_B \cdots$$

In particular, for each *fixed* $n \geq 1$, the isomorphism relation $\cong_n$ on the space of torsion-free abelian groups of rank n is not countable universal. Of course, this strongly suggests that the isomorphism relation $\bigsqcup_{n\geq 1} \cong_n$ on the space $\bigsqcup_{n\geq 1} R(\mathbb{Q}^n)$ of torsion-free abelian groups of *finite* rank is also not countable universal. In the remainder of this section, we shall confirm that this is indeed the case.

Theorem 6.1. *The isomorphism relation on the space of torsion-free abelian groups of finite rank is not countable universal.*

Of course, Theorem 6.1 would follow trivially from the previous results if it were known that a smooth disjoint union of countably many non-universal countable Borel equivalence relations was also non-universal. Unfortunately, as mentioned earlier, this question remains open even for the case $E_1 \sqcup E_2$ of two non-universal relations. However, the following weaker statement is an immediate consequence of Theorem 1.2(c) and Corollary 3.10.

Proposition 6.2. *If E_n, $n \in \mathbb{N}$, are essentially free countable Borel equivalence relations, then $\bigsqcup_{n\in\mathbb{N}} E_n$ is not countable universal.*

□

It is well-known that $\cong_1$ is hyperfinite and hence is essentially free. It is conceivable that Król's analysis [18] might be enough to prove that $\cong_2$ is also essentially free. (Cf. Thomas [35].) However, the following question appears to be very difficult when $n \geq 3$.

Question 6.3. Let $n \geq 2$. Is the isomorphism relation $\cong_n$ on the space of torsion-free abelian groups of rank n essentially free?

Roughly speaking, the following proof of Theorem 6.1 will depend upon the weaker result that each $\cong_n$ is (hyperfinite)-by-(essentially free). From now on, let S be a *suitably chosen* countable group and let

$$G = SL_3(\mathbb{Z}) \times S.$$

(We will give a more precise definition of S at the appropriate point in the proof.) Let E_G be the orbit equivalence relation of the action of G on $((2)^G, \mu)$ and suppose that

$$f : (2)^G \to \bigsqcup_{n \geq 1} R(\mathbb{Q}^n)$$

is a Borel reduction from E_G to the isomorphism relation $\bigsqcup_{n\geq 1} \cong_n$. After deleting a nullset of $(2)^G$ if necessary, we can suppose that

$$f : (2)^G \to R(\mathbb{Q}^n)$$

for some fixed $n \geq 1$. Unfortunately, we can not define a corresponding cocycle at this point, since action of $GL_n(\mathbb{Q})$ is not free. In fact, for each $A \in R(\mathbb{Q}^n)$, the stabilizer of A in $GL_n(\mathbb{Q})$ is precisely the automorphism group $\mathrm{Aut}(A)$ of A. To get around this difficulty, we shall shift our focus from the isomorphism relation on $R(\mathbb{Q}^n)$ to the coarser quasi-isomorphism relation.

Definition 6.4. Let A, $B \in R(\mathbb{Q}^n)$. Then:

(a) A and B are said to be *quasi-equal*, written $A \approx_n B$, iff $A \cap B$ has finite index in both A and B.

(b) A and B are said to be *quasi-isomorphic*, written $A \sim_n B$, if there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \approx_n B$.

The next result shows that we do not lose too much information in passing from the isomorphism relation to the quasi-isomorphism relation.

Theorem 6.5 (Thomas [36])**.** *The quasi-equality relation $\approx_n$ is a hyperfinite countable Borel equivalence relation.*

For each $A \in R(\mathbb{Q}^n)$, let $[A]$ be the $\approx_n$-class containing A. We shall consider the induced action of $GL_n(\mathbb{Q})$ on the set

$$X = \{[A] \mid A \in R(\mathbb{Q}^n)\}$$

of $\approx_n$-classes. (Of course, since $\approx_n$ is nonsmooth, it follows that X is *not* a standard Borel space. Fortunately, this will not lead to any difficulties.) In order to describe the setwise stabilizer in $GL_n(\mathbb{Q})$ of each $\approx_n$-class $[A]$, it is first necessary to introduce the notions of a quasi-endomorphism and a quasi-automorphism.

Definition 6.6. Let $A \in R(\mathbb{Q}^n)$. Then:

(a) The ring of *quasi-endomorphisms* of A is defined to be

$$\mathrm{QE}(A) = \{\varphi \in \mathrm{Mat}_n(\mathbb{Q}) \mid (\exists m \geq 1)\, m\varphi \in \mathrm{End}(A)\}.$$

(b) A linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-automorphism* of A iff φ is a unit of the ring $\mathrm{QE}(A)$. The group of quasi-automorphisms of A is denoted by $\mathrm{QAut}(A)$.

It is easily checked that $\mathrm{QE}(A)$ is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$. In particular, it follows that there are only countably many possibilities for $\mathrm{QAut}(A)$.

Lemma 6.7 (Thomas [36]). *If $A \in R(\mathbb{Q}^n)$, then $\mathrm{QAut}(A)$ is the setwise stabilizer of $[A]$ in $GL_n(\mathbb{Q})$.*

For each $x \in (2)^G$, let $A_x = f(x) \in R(\mathbb{Q}^n)$. Since there are only countably possibilities for the group $\mathrm{QAut}(A_x)$, there exists a *fixed* subgroup $L \leqslant GL_n(\mathbb{Q})$ and a Borel subset $X \subseteq (2)^G$ with $\mu(X) > 0$ such that $\mathrm{QAut}(A_x) = L$ for all $x \in X$. Since G acts ergodically on $(\,(2)^G, \mu\,)$, it follows that $\mu(G \cdot X) = 1$. In order to simplify notation, we shall assume that $G \cdot X = (2)^G$. After slightly adjusting f if necessary, we can suppose that $\mathrm{QAut}(A_x) = L$ for all $x \in (2)^G$. (More precisely, let $c : X \to X$ be a Borel function such that $c(x)\, E_G\, x$ and $c(x) \in X$ for all $x \in (2)^G$. Then we can replace f with $f' = f \circ c$.)

Now suppose that x, $y \in (2)^G$ and that $x\, E_G\, y$. Then $A_x \cong_n A_y$ and so there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A_x) = A_y$. Notice that

$$\varphi L \varphi^{-1} = \varphi\, \mathrm{QAut}(A_x) \varphi^{-1} = \mathrm{QAut}(\varphi(A_x)) = \mathrm{QAut}(A_y) = L$$

and so $\varphi \in N = N_{GL_n(\mathbb{Q})}(L)$. Clearly we also have that $\varphi \cdot [A_x] = [A_y]$. Furthermore, by Lemma 6.7, for each $x \in (2)^G$, the stabiliser of $[A_x]$ in $GL_n(\mathbb{Q})$ is $\mathrm{QAut}(A_x) = L$. Let $H = N/L$ and for each $\varphi \in N$, let $\overline{\varphi} = \varphi L \in H$. Then we can define a Borel cocycle

$$\alpha : G \times (2)^G \to H$$

by setting

$$\alpha(g,x) = \text{ the unique } \overline{\varphi} \in H \text{ such that } \varphi \cdot [A_x] = [A_{g\cdot x}].$$

Now let S be a countable simple nonamenable group which does not embed into any of the countably many possibilities for H. (To see that such a group exists, let T be a finitely generated nonamenable group which does not embed into any of the countably many possibilities for H and let S be a countable simple group which into which T embeds.) Applying Theorem 5.3, after deleting a nullset of $(2)^G$ and slightly adjusting f if necessary, we can suppose that

$$\alpha : G = SL_3(\mathbb{Z}) \times S \to H$$

is a group homomorphism. By the choice of S, we must have that $S \leqslant \ker \alpha$. Hence if $g \in S$, then

$$[A_{g\cdot x}] = [A_x] \quad \text{ for } \mu\text{-a.e. } x \in (2)^G.$$

In other words, after deleting a nullset of $(2)^G$, the map $f : (2)^G \to R(\mathbb{Q}^n)$ is a Borel homomorphism from the S-action on $(2)^G$ to the hyperfinite quasi-equality $\approx_n$-relation. By Hjorth-Kechris [12, Theorem A4.1], since S is nonamenable, the S-action on $(2)^G$ is E_0-ergodic and hence μ-almost all $x \in (2)^G$ are mapped to a single $\approx_n$-class, which is a contradiction. This completes the proof of Theorem 6.1.

7. Quasi-finite groups

In this section, we shall study the orbit equivalence relations arising from the (not necessarily free) Borel actions of quasi-finite groups. In particular, we shall show that no such countable Borel equivalence relation is universal. But first we shall present a proof of Proposition 3.15, which makes use of the existence of a suitable uncountable family of simple quasi-finite groups.

Definition 7.1. An infinite group G is said to be *quasi-finite* iff every proper subgroup of G is finite.

It is easily shown that every abelian quasi-finite group is isomorphic to a quasi-cyclic group C_{p^∞} for some prime p. (See Ol'shanskii [27, Theorem 7.5].) However, it was a long outstanding problem whether there existed a nonabelian quasi-finite group. This problem was finally solved by Ol'shanskii in his celebrated papers [24, 25]. A clear account of this work can be found in Ol'shanskii [27].

The following result is essentially a restatement of Ol'shanskii [27, Theorem 28.6]. (While Ol'shanskii does not state his result in terms of standard Borel spaces, it is easily checked that the map $x \mapsto T_x$ is Borel.)

Theorem 7.2 (Ol'shanskii [27]). *Let $\mathcal{P}$ be the standard Borel space of all strictly increasing sequences $x = \langle p_n \mid n \in \mathbb{N} \rangle$ of primes such that $p_0 > 10^{75}$. Then there exists a Borel family $\{T_x \mid x \in \mathcal{P}\}$ of 2-generator groups such that for every $x = \langle p_n \mid n \in \mathbb{N} \rangle \in \mathcal{P}$, the following conditions are satisfied:*

(a) *T_x contains a cyclic subgroup of order p_n for each $n \in \mathbb{N}$.*
(b) *Every nontrivial proper subgroup of T_x is cyclic of order p_n for some $n \in \mathbb{N}$.*
(c) *T_x is simple.*

□

Clearly Proposition 3.15 is an immediate consequence of the following result.

Proposition 7.3. *For each $x \in \mathcal{P}$, let $G_x = SL_3(\mathbb{Z}) \times T_x$. Then the Borel family $\{G_x \mid x \in \mathcal{P}\}$ of finitely generated groups satisfies the following conditions:*

(i) *If $x \neq y$, then G_x and G_y are not isomorphic up to finite kernels.*
(ii) *If $x \neq y$, then G_x does not virtually embed into G_y.*

Proof. Since $SL_3(\mathbb{Z})$ has no nontrivial finite normal subgroups and each T_x is an infinite simple group, it follows that each group G_x also has no nontrivial finite normal subgroups. Hence it is enough to prove that if $x \neq y$, then G_x does not embed into G_y. Suppose that

$$\pi : G_x = SL_3(\mathbb{Z}) \times T_x \to SL_3(\mathbb{Z}) \times T_y = G_y$$

is an embedding. Let $p : G_y \to SL_3(\mathbb{Z})$ be the canonical projection and consider the homomorphism

$$\varphi = p \circ \pi : SL_3(\mathbb{Z}) \times T_x \to SL_3(\mathbb{Z}).$$

By Wehrfritz [38, Corollary 4.9], every finitely generated periodic linear group is finite. It follows that $T_x \leqslant \ker \varphi$ and hence π embeds T_x into T_y. But then $\pi(T_x)$ is an infinite proper subgroup of T_y, which is a contradiction. □

In the remainder of this section, we shall study the orbit equivalence relations arising from arbitrary (not necessarily free) Borel actions of quasi-finite groups. This study is motivated by the following problem.

Problem 7.4. Classify the countable groups G for which there exists a standard Borel G-space X such that E_G^X is countable universal.

By Dougherty-Jackson-Kechris [5], if G contains a nonabelian free subgroup, then the orbit equivalence relation arising from the shift action of G on 2^G is countable universal. (Of course, it is essential here to consider the action on the entire space 2^G rather than on just the free part $(2)^G$ of the action.) On the other hand, by Jackson-Kechris-Louveau [14, Section 2], if G is a countable amenable group and X is a standard Borel G-space X, then E_G^X is never countable universal. Of course, this raises the possibility of a "dynamic" version of the von Neumann Conjecture that every countable group is either amenable or else contains a nonabelian free subgroup. (The original von Neumann conjecture, which is actually due to Day, was disproved by Ol'shanskii [26] in 1980. For another equally implausible "dynamic" version due to Gaboriau, see Kechris-Miller [16, Problem 28.14].) The following result constitutes a very modest contribution to this presumably very difficult problem.

Theorem 7.5. *Suppose that G is a quasi-finite group and that X is a standard Borel G-space. Then E_G^X is not countable universal.*

Remark 7.6. It is an open question whether every nonabelian quasi-finite group is nonamenable. However, the results of Ol'shanskii [28] imply that there exist nonabelian quasi-finite groups with the Kazhdan property and these groups are certainly nonamenable.

Of course, in the proof of Theorem 7.5, we can restrict our attention to the case when G is nonabelian. In this case, it is well-known that G is almost simple. (As I have not been able to find a reference for this result, I have included the following easy proof.)

Proposition 7.7. *If G is a nonabelian quasi-finite group, then $Z(G)$ is finite and $G/Z(G)$ is simple.*

Proof. It is clear that $Z(G)$ is finite. In order to prove that $G/Z(G)$ is simple, it is enough to show that $Z(G)$ is the unique maximal proper normal subgroup of G. To see this, suppose that N is a proper normal subgroup and let $x \in N$. Then

$$[G : C_G(x)] = |x^G| \leq |N| < \infty$$

and so $C_G(x) = G$; i.e. $x \in Z(G)$. □

As we shall soon see, Theorem 7.5 is an easy consequence of the following result.

Lemma 7.8. *Suppose that G is a simple quasi-finite group and that X is a standard Borel G-space. Let*

$$Y = \{x \in X \mid \text{ There exists } 1 \neq g \in G \text{ such that } g \cdot x = x\}$$

be the non-free part of the action. Then $E_G^X \restriction Y$ is smooth.

Proof. If $Z = \{x \in Y \mid g \cdot x = x \text{ for all } g \in G\}$, then $E_G^X \restriction Z$ is clearly smooth and so we can suppose that $Z = \emptyset$. Fix an element $F_{\mathcal{C}}$ of each of the countably many conjugacy classes $\mathcal{C}$ of nontrivial finite subgroups of G.

If $x \in Y$, then $G_x = \{g \in G \mid g \cdot x = x\}$ is a nontrivial finite subgroup of G. Let $\mathcal{C}$ be the conjugacy class containing G_x and define

$$\pi(x) = \{y \in G \cdot x \mid G_y = F_{\mathcal{C}}\}.$$

We claim that $\pi(x)$ is a nonempty finite subset of the orbit $G \cdot x$. To see this, first choose $g \in G$ such that $gG_xg^{-1} = F_{\mathcal{C}}$ and let $y = g \cdot x$. Then $G_y = gG_xg^{-1} = F_{\mathcal{C}}$ and so $y \in \pi(x)$. Next suppose that $y, z \in \pi(x)$ and let $g \cdot y = z$. Since

$$gF_{\mathcal{C}}g^{-1} = gG_yg^{-1} = G_z = F_{\mathcal{C}},$$

it follows that $g \in N_G(F_{\mathcal{C}})$. Since G is simple, $N_G(F_{\mathcal{C}})$ is a proper subgroup of G and so $|N_G(F_{\mathcal{C}})| < \infty$. Hence $\pi(x)$ is finite. Clearly if $G \cdot x = G \cdot y$, then $\pi(x) = \pi(y)$. Thus the Borel map $\pi : Y \to Y^{<\omega}$ witnesses that $E_G^X \restriction Y$ is smooth. □

Proof of Theorem 7.5. By Jackson-Kechris-Louveau [14], if G is abelian, then E_G^X is not countable universal. Hence we can suppose that $Z(G)$ is finite and that $\overline{G} = G/Z(G)$ is simple. Since $Z(G)$ is finite, it follows that the orbit space

$$\overline{X} = \{Z(G) \cdot x \mid x \in X\}$$

is a standard Borel space. It is easily checked that the map $x \mapsto Z(G) \cdot x$ is a Borel bireduction between E_G^X and the orbit equivalence relation $E_{\overline{G}}^{\overline{X}}$ of the naturally induced $\overline{G}$-action on $\overline{X}$. Hence we can suppose that G is simple. But then Lemma 7.8 implies that E_G^X is essentially free and hence is not countable universal. $\square$

Appendix A

In this appendix, we shall present a proof of the following result. The argument is an extension of the proof of Thomas [37, Theorem 5.1], which in turn was based on the ideas of Gefter-Golodets [10].

Theorem A.1. *There exists a pair E, F of countable Borel equivalence relations on standard Borel spaces X, Y such that:*

(a) *there exists a unique E-invariant probability measure on X;*
(b) *E is not F-ergodic; and*
(c) *$E \not\leq_B^w F$.*

From now on, let $\mathbb{P}_0$ be the set of odd primes p such that $p \equiv 2 \mod 3$. It is well-known that these are exactly the odd primes p such that the ring $\mathbb{Z}_p$ of p-adic integers does not contain a primitive third root of unity. (For example, see Robert [32, Section I.6.7].) It is also well-known that $\mathbb{P}_0$ is an infinite set of primes. (For example, see Ireland-Rosen [13].) However, for our purposes, it is enough that there are at least 2 such primes. For each nonempty subset $J \subseteq \mathbb{P}_0$, let

$$K(J) = \prod_{p \in J} SL_3(\mathbb{Z}_p).$$

Then $K(J)$ is a compact second countable group and we can regard $\Gamma = SL_3(\mathbb{Z})$ as a subgroup of $K(J)$ via the diagonal embedding. Let μ_J be the Haar probability measure on $K(J)$ and let E_J be the orbit equivalence relation arising from the free action of Γ on $K(J)$ via left translations. By the Strong Approximation Theorem [29, Theorem 7.12], Γ is a dense subgroup of $K(J)$ and hence μ_J is the unique E_J-invariant probability measure on $K(J)$.

Let J_0 and J_1 be nonempty subsets of $\mathbb{P}_0$ such that $J_1 \subsetneq J_0$. We shall show that the countable Borel equivalence relations $E = E_{J_0}$, $F = E_{J_1}$ on the standard Borel spaces $X = K(J_0)$, $Y = K(J_1)$ satisfy the conditions of Theorem A.1. We have already noted that condition (a) holds. In order to see that condition (b) holds, let

$\pi : K(J_0) \to K(J_1)$ be the canonical surjective homomorphism. Then π is a Borel homomorphism from E to F such that $\mu_{J_0}(\pi^{-1}(y)) = 0$ for all $y \in K(J_1)$. Hence E is not F-ergodic.

Finally, in order to see that condition (c) holds, suppose that $f : K(J_0) \to K(J_1)$ is a countable-to-one Borel homomorphism from E to F. Then, applying Thomas [37, Theorem 4.4], there exist

- subgroups $\Lambda_0, \Lambda_1 \leqslant \Gamma$ with $[\Gamma : \Lambda_0] = [\Gamma : \Lambda_1] < \infty$,
- ergodic components Z_i, $i = 0, 1$, for the action of Λ_i on $K(J_i)$,
- a Borel map $\tilde{f} : Z_0 \to Z_1$, and
- an isomorphism $\varphi : \Lambda_0 \to \Lambda_1$

such that the following conditions are satisfied:

(i) $\tilde{f}_*(\mu_{J_0})_{Z_0} = (\mu_{J_1})_{Z_1}$.
(ii) $\tilde{f}(\gamma \cdot x) = \varphi(\gamma) \cdot \tilde{f}(x)$ for all $\gamma \in \Lambda_0$ and $x \in Z_0$.
(iii) $\tilde{f}(x) \in \Gamma \cdot f(x)$ for all $x \in Z_0$.

In condition (i), $(\mu_{J_i})_{Z_i}$ denotes the probability measure on Z_i defined by

$$(\mu_{J_i})_{Z_i}(A) = \mu_{J_i}(A)/\mu_{J_i}(Z_i).$$

Notice that condition (iii) implies that $\tilde{f}$ is also a countable-to-one map. For each $0 \leq i \leq 1$, let H_i be the closure of Λ_i in $K(J_i)$ and let $\mu_i = (\mu_{J_i})_{Z_i}$. Then H_i is an open subgroup of $K(J_i)$; and by Thomas [37, Lemma 2.2], we can suppose that $Z_i \subseteq H_i$ and that μ_i is the Haar probability measure on H_i.

For each $t \in H_0$, consider the Borel map $h_t : Z_0 \to Z_1$ defined by

$$h_t(x) = \tilde{f}(x)^{-1}\tilde{f}(xt) \quad \mu_0\text{-a.e. } x \in Z_0.$$

Then for all $\gamma \in \Lambda_0$, we have that for μ_0-a.e. $x \in Z_0$,

$$\begin{aligned} h_t(\gamma \cdot x) &= \tilde{f}(\gamma \cdot x)^{-1}\, \tilde{f}(\gamma \cdot xt) \\ &= \tilde{f}(x)^{-1}\, \varphi(\gamma)^{-1}\, \varphi(\gamma)\, \tilde{f}(xt) \\ &= h_t(x). \end{aligned}$$

Since Λ_0 acts ergodically on (Z_0, μ_0), there exists an element $\theta(t) \in H_1$ such that

$$h_t(x) = \theta(t) \quad \mu_0\text{-a.e. } x \in Z_0.$$

In other words, we have that for all $t \in H_0$,

$$\tilde{f}(xt) = \tilde{f}(x)\,\theta(t) \quad \mu_0\text{-a.e. } x \in Z_0.$$

By Kechris [15, Theorem 17.25], the map $\theta : H_0 \to H_1$ is Borel. Furthermore, an easy calculation shows that θ is a group homomorphism. Hence, applying Kechris [15, Theorem 9.10], it follows that $\theta : H_0 \to H_1$ is a continuous homomorphism. In particular, $N = \ker\theta$ is a closed subgroup of H_0. Let ν be the Haar probability measure on N. By Fubini's Theorem, we have that for μ_0-a.e. $x \in Z_0$,

$$\tilde{f}(xt) = \tilde{f}(x) \quad \nu\text{-a.e. } t \in N.$$

As $\tilde{f}$ is countable-to-one, this implies that N is countable and hence is finite. It is well-known that if p is an arbitrary prime, then the finite normal subgroups of $SL_3(\mathbb{Z}_p)$ are contained in its center, which consists of the scalar matrices $d\,I$ with $d^3 = 1$. In particular, if $p \in \mathbb{P}_0$, since $\mathbb{Z}_p$ does not contain a primitive third root of unity, it follows that $SL_3(\mathbb{Z}_p)$ has no nontrivial finite normal subgroups. This easily implies that the same is true of $K(J_0)$; and since $[K(J_0) : H_0] < \infty$, the same is also true of H_0. Thus θ is an injective homomorphism. Applying Fubini's Theorem once again, there exists $x_0 \in Z_0$ such that

$$\tilde{f}(x_0 t) = \tilde{f}(x_0)\,\theta(t) \quad \mu_0\text{-a.e. } t \in H_0.$$

Since $\tilde{f}_*\mu_0 = \mu_1$, we must have that

$$\mu_1(\theta(H_0)) = \mu_1(\tilde{f}(Z_0)) = 1$$

and so θ is also surjective. Since H_0, H_1 are compact Hausdorff, it follows that θ is a homeomorphism. But this contradicts Gefter-Golodets [10, Lemma A.6] which says that the groups $K(J_0)$ and $K(J_1)$ do not contain open subgroups which are isomorphic as topological groups. This completes the proof of Theorem A.1.

References

[1] S. Adams, *Containment does not imply Borel reducibility*, in: S. Thomas (Ed.), *Set Theory: The Hajnal Conference*, DIMACS Series vol. 58, American Mathematical Society, 2002, pp. 1–23.

[2] S. R. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), 909–943.

[3] A. Andretta, R. Camerlo and G. Hjorth, *Conjugacy equivalence relations on subgroups*, Fund. Math. **167** (2001), 189–212.

[4] M. B. Bekka and M. Mayer, *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces*, London Math. Soc. Lecture Notes Series **269**, Cambridge University Press, 2000.

[5] R. Dougherty, S. Jackson and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193–225.

[6] R. Dougherty and A. S. Kechris, *How many Turing degrees are there?* in: Computability Theory and Its Applications, Current Trends and Open Problems, Ed. by P.A. Cholak et al., Contemp. Math. **257**, 83-95, Amer. Math. Soc., 2000

[7] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[8] A. Furman, *Orbit equivalence rigidity*, Ann. Math. **150** (1999), 1083–1108.

[9] A. Furman, *On Popa's cocycle superrigidity theorem*, preprint (2006).

[10] S. L. Gefter and V. Ya. Golodets, *Fundamental groups for ergodic actions and actions with unit fundamental groups*, Publ. RIMS, Kyoto Univ. **24** (1988), 821–847.

[11] G. Hjorth, *Around nonclassifiability for countable torsion-free abelian groups*, in Abelian groups and modules (Dublin, 1998), Trends Math., Birkhäuser, Basel, 1999, 269–292.

[12] G. Hjorth and A.S. Kechris, *Rigidity theorems for actions of product groups and countable Borel equivalence relations*, Mem. Amer. Math. Soc. **177**, no. 833, 2005.

[13] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics **84**, Springer-Verlag, 1982.

[14] S. Jackson, A.S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Logic **2** (2002), 1–80.

[15] A. S. Kechris *Classical Descriptive Set Theory*, Graduate Texts in Mathematics **156**, Springer-Verlag, 1995.

[16] A.S. Kechris and B.D. Miller, *Topics in Orbit Equivalence*, Lecture Notes in Mathematics, Springer, Vol. 1852, 2004.

[17] A.S. Kechris and Y.N. Moschovakis, Cabal Seminar 76-77, Lecture Notes in Mathematics **689**, Springer Verlag, 1978.

[18] M. Król, *The automorphism groups and endomorphism rings of torsion-free abelian groups of rank two*, Dissertationes Math. Rozprawy Mat. **55** (1967), 1–76.

[19] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Mathematics **125**, Birkhäuser, 1994.

[20] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Erg. der Math. und ihrer Grenz., **17**, Springer-Verlag, 1991.

[21] B.D. Miller, *Borel equivalence relations and everywhere faithful actions of free products*, preprint (2006).

[22] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology*, Ann. Math. **164** (2006), 825-878

[23] B. H. Neumann, *Some remarks on infinite groups*, J. London Math. Soc. **12** (1937), 120–127.

[24] A. Yu. Ol'shanskii, *Infinite groups with cyclic subgroups*, Dokl. Akad. Nauk SSSR Ser. Mat. **43** (1979), 785–787.

[25] A. Yu. Ol'shanskii, *An infinite group with subgroups of prime order*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 309–321.

[26] A. Yu. Ol'shanskii, *On the question of the existence of an invariant mean on a group*, Uspekhi Mat. Nauk **35** (1980), 199–200.

[27] A. Yu. Ol'shanskii, *Geometry of defining relations in groups*, Mathematics and Its Applications (Soviet Series), vol. 70 (Kluwer Academic Publishers, Dordrecht, 1991).

[28] A. Yu. Ol'shanskii, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Internat. J. Algebra Comput. **3** (1993), 365–409.

[29] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, 1994.

[30] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, preprint (2005).

[31] S. Popa, *On the superrigidity of malleable actions with spectral gap*, preprint (2006).

[32] A. M. Robert, *A Course in p-adic Analysis*, Graduate Texts in Mathematics **198**, Springer-Verlag, 2000.

[33] R. Steinberg, *Some consequences of the elementary relations in SL_n*, in Finite Groups - Coming Of Age, Contemporary Math. **45**, Amer. Math. Soc. (1985), 335–350.

[34] S. Thomas, *Some applications of superrigidity to Borel equivalence relations*, in: S. Thomas (Ed.), *Set Theory: The Hajnal Conference*, DIMACS Series vol. 58, American Mathematical Society, 2002, pp. 129–134.

[35] S. Thomas, *On the complexity of the classification problem for torsion-free abelian groups of rank two*, Acta Math. **189** (2002), 287–305.

[36] S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*, J. Amer. Math. Soc. **16** (2003), 233–258.

[37] S. Thomas, *Superrigidity and countable Borel equivalence relations*, Ann. Pure Appl. Logic **120** (2003), 237–262.

[38] B.A.F. Wehrfritz, *Infinite Linear Groups*, Springer-Verlag, Berlin, 1973.

[39] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, 1984.

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