ON THE E_0-EXTENSIONS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

SIMON THOMAS

ABSTRACT. We initiate the study of the E_0-extensions of Borel actions $\Gamma \curvearrowright X$ of countable groups Γ on standard Borel spaces X.

1. Introduction

Let $\Gamma \curvearrowright X$ be a Borel action of a countable group Γ on a standard Borel space X and let E_Γ^X be the corresponding orbit equivalence relation. Suppose that E is a countable Borel equivalence relation on X such that:

- E is Borel isomorphic to the Vitali equivalence relation E_0 on $2^{\mathbb{N}}$;
- Γ normalizes E; i.e. the action $\Gamma \curvearrowright X$ permutes the E-classes.

Then the corresponding E_0-*extension* of E_Γ^X is the countable Borel equivalence relation $E \rtimes E_\Gamma^X$ on X defined by

$$x\,(E \rtimes E_\Gamma^X)\,y \quad \Longleftrightarrow \quad \text{there exists } \gamma \in \Gamma \text{ such that } \gamma \cdot x \; E \; y.$$

This definition includes the possibility that $E \subseteq E_\Gamma^X$, in which case we will say that $E \rtimes E_\Gamma^X = E_\Gamma^X$ is a *trivial* E_0-extension. Thus E_Γ^X is a trivial E_0-extension of itself precisely when there exists a subequivalence relation $E \subseteq E_\Gamma^X$ with $E \cong E_0$, which is normalized by the Borel action $\Gamma \curvearrowright X$. (It is usually a highly nontrivial question to classify the normal subequivalence relations of a countable Borel equivalence relation. See Feldman-Sutherland-Zimmer [5, Section 4] or Bowen [2].)

Examples of E_0-extensions arise naturally in the analysis of the complexity of the quasi-isomorphism relation on the space of torsion-free abelian groups of finite rank. (See Thomas [18].) In this paper, we will consider the problem of determining the

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relative complexity of E_Γ^X and $E \rtimes E_\Gamma^X$ for various naturally occuring E_0-extensions $E \rtimes E_\Gamma^X$; and we will point out some fundamental open questions.

Of course, the most basic example occurs when $\Gamma = 1$, $X = 2^{\mathbb{N}}$ and $E = E_0$. In this case, $E_\Gamma^X = \mathrm{Id}_{2^{\mathbb{N}}}$ and $E \rtimes E_\Gamma^X = E_0$, and so $E_\Gamma^X <_B E \rtimes E_\Gamma^X$. There also exist many natural examples of nontrivial E_0-extensions such that $E_\Gamma^X \sim_B E \rtimes E_\Gamma^X$.

Definition 1.1. If Γ is a countably infinite group, then E_Γ denotes the orbit equivalence relation of the usual shift action $\Gamma \curvearrowright 2^\Gamma$.

If Γ is a countably infinite group, then we will also denote the corresponding Vitali equivalence relation on 2^Γ by E_0. Suppose that Γ contains a free nonabelian subgroup. Then, by Dougherty-Jackson-Kechris [3, Section 1], E_Γ is a universal countable Borel equivalence relation.

Theorem 1.2. *If Γ is a countably infinite group which contains a free nonabelian subgroup, then $E_0 \rtimes E_\Gamma$ is a universal countable Borel equivalence relation, and hence $E_\Gamma \sim_B E_0 \rtimes E_\Gamma$.*

Initially it might seem reasonable to expect that $E_\Gamma^X \leq_B E \rtimes E_\Gamma^X$ for every E_0-extension. However, it turns out that there is a naturally occuring counterexample. Fix some integer $n \geq 3$ and let $\mathbb{P}$ be the set of primes. For each $p \in \mathbb{P}$, let $V(n,p) = \mathbb{F}_p^n$ be the n-dimensional vector space over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and let $g \mapsto g_p$ be the group homomorphism $SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_p)$ induced by the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Let

$$\Gamma = SL_n(\mathbb{Z}) \curvearrowright V = \prod_{p \in \mathbb{P}} V(n,p)$$

be the Borel action defined by $g \cdot (v_p) = (g_p \cdot v_p)$. Let E_0^V be the countable Borel equivalence relation defined on V by

$$(v_p) \, E_0^V \, (w_p) \quad \Longleftrightarrow \quad v_p = w_p \text{ for all but finitely many } p \in \mathbb{P}.$$

Then E_0^V is a uniquely ergodic, aperiodic, hyperfinite Borel equivalence relation; and it follows that E_0^V is Borel isomorphic to E_0. (For example, see Dougherty-Jackson-Kechris [3, Theorem 9.1].) Also it is clear that the action $\Gamma \curvearrowright V$ permutes the E_0^V-classes.

Theorem 1.3. *E_Γ^V and $E_0^V \rtimes E_\Gamma^V$ are incomparable with respect to Borel reducibility.*

By Thomas [17, Theorem 5.5], there exists a pair of countable Borel equivalence relations $E \subseteq F$ on a standard Borel space Z such that $F <_B E$. However, it is not known whether such examples exist in the context of E_0-extensions.

Question 1.4. Does there exist an E_0-extension $E \rtimes E_\Gamma^X$ such that $E \rtimes E_\Gamma^X <_B E_\Gamma^X$?

It is also natural to consider the question of which countable Borel equivalence relations admit E_0-extensions. (Concrete instances of this question arise in the analysis of the relative complexity of the isomorphism and quasi-isomorphism relations on the space of torsion-free abelian groups of finite rank.) Here it is necessary to distinguish between group actions and countable Borel equivalence relations.

Definition 1.5. If $\Gamma \curvearrowright X$ is a Borel action of a countable group Γ on a standard Borel space X, then $\Gamma \curvearrowright X$ *admits an E_0-extension* if there exists a countable Borel equivalence relation E on X such that $E \cong E_0$ and Γ normalizes E.

Definition 1.6. If F is a countable Borel equivalence relations on a standard Borel space X, then F *admits an E_0-extension* if there exists a Borel action $\Gamma \curvearrowright X$ of a countable group Γ such that $F = E_\Gamma^X$ and $\Gamma \curvearrowright X$ admits an E_0-extension.

There is an obvious necessary condition for a countable Borel equivalence relation to admit an E_0-extension.

Proposition 1.7. *If a countable Borel equivalence relation F on a standard Borel space X admits an E_0-extension, then there exists an F-invariant nonatomic probability measure on X.*

Proof. Let $\Gamma \curvearrowright X$ be a Borel action of a countable group Γ such that $F = E_\Gamma^X$ and such that there exists a countable Borel equivalence relation E on X such that:

- E is Borel isomorphic to the Vitali equivalence relation E_0 on $2^{\mathbb{N}}$;
- Γ normalizes E.

Since $E \cong E_0$, it follows that there exists a unique E-invariant probability measure ν on X and that ν is nonatomic. Since Γ acts as a group of automorphisms of (X, E), it follows that ν is Γ-invariant, and thus ν is F-invariant. □

It is currently unknown whether or not the converse of Proposition 1.7 also holds.

Question 1.8. Does there exist a countable Borel equivalence relation F with an F-invariant nonatomic probability measure such that F does *not* admit an E_0-extension?

On the other hand, there exist many examples of Borel actions $\Gamma \curvearrowright X$ of countable groups Γ on standard Borel spaces X such that $\Gamma \curvearrowright X$ does not admit an E_0-extension.

Theorem 1.9. *If F is an aperiodic nonhyperfinite countable Borel equivalence relation on a standard Borel space X, then there exists a Borel action $\Gamma \curvearrowright X$ of a countable group Γ such that $F = E_\Gamma^X$ and such that $\Gamma \curvearrowright X$ does not admit an E_0-extension.*

If F is a countable Borel equivalence relation on a standard Borel space X, then $[\,F\,]$ denotes the corresponding Borel full group; i.e. the group of all Borel automorphisms of $f : (\,X, F\,) \to (\,X, F\,)$. The proof of Theorem 1.9 will make use of the following result. (Recall that a permutation group G on an infinite set Ω is said to be *highly transitive* if G acts k-transitively on Ω for every $k \geq 1$.)

Lemma 1.10. *If F is an aperiodic countable Borel equivalence relation on a standard Borel space X, then there exists a countable subgroup $\Gamma \leqslant [\,F\,]$ such that $F = E_\Gamma^X$ and such that:*

- (i) *Γ acts highly transitively on each F-class;*
- (ii) *$\Gamma_x \neq \Gamma_y$ for all $x \neq y \in X$.*

Sketch proof of Lemma 1.10. Clearly we can suppose that X is an uncountable standard Borel space. Then X is Borel isomorphic to $\mathbb{R}$; and it follows that we can express

$$X^2 \smallsetminus \{\, (x,x) \mid x \in X \,\} = \bigcup_{\ell \in \mathbb{N}} (\, A_\ell \times B_\ell \,),$$

where A_ℓ, B_ℓ are Borel subsets of X such that:

- $A_\ell \cap B_\ell = \emptyset$ for all $\ell \in \mathbb{N}$;
- if $(\, x_1, \cdots, x_k, x_{k+1}, x_{k+2}\,)$ is a finite sequence of distinct elements of X, then there exists $\ell \in \mathbb{N}$ such that $x_{k+1} \in A_\ell$, $x_{k+2} \in B_\ell$, and $x_m \notin A_\ell \cup B_\ell$ for all $1 \leq m \leq k$.

Now the proof of Kechris-Miller [10, Theorem 1.3][1] yields the existence of a sequence of Borel bijections $(g_n \mid g \in \mathbb{N})$ such that whenever $(x_1, \cdots, x_k, x_{k+1}, x_{k+2})$ is a finite sequence of distinct elements of X with $x_{k+1} \, F \, x_{k+2}$, then there exist $n \in \mathbb{N}$ such that $g_n(x_{k+1}) = x_{k+2}$ and $g_n(x_m) = x_m$ for all $1 \leq m \leq k$. Let $\Gamma \leqslant [\, F \,]$ be the subgroup generated by $\{ g_n \mid n \in \mathbb{N} \}$. Then clearly $F = E_\Gamma^X$; and an easy inductive argument shows that Γ acts k-transitively on every F-class for each $k \geq 1$. □

Proof of Theorem 1.9. Applying Lemma 1.10, let $\Gamma \leqslant [\, F \,]$ be a countable subgroup such that $F = E_\Gamma^X$ and such that:

(i) Γ acts highly transitively on each F-class;
(ii) $\Gamma_x \neq \Gamma_y$ for all $x \neq y \in X$.

Note that (i) implies that Γ acts primitively[2] on each F-class; and it follows that Γ_x is a maximal proper subgroup of Γ for each $x \in X$.

Suppose that E is a countable Borel equivalence relation on X such that $E \cong E_0$ and Γ normalizes E. Let $x \in X$. Then, since Γ acts primitively on $[\, x \,]_F$, it follows that either $[\, x \,]_F \subseteq [\, x \,]_E$ or else $E \restriction [\, x \,]_F$ is the equality relation on $[\, x \,]_F$.

Applying Dougherty-Jackson-Kechris [3, Proposition 5.2], if $[\, x \,]_F \subseteq [\, x \,]_E$ for all $x \in X$, then F is hyperfinite, which is a contradiction. Hence there exists $x \in X$ such that $E \restriction [\, x \,]_F$ is the equality relation on $[\, x \,]_F$. Let $y \in [\, x \,]_E \smallsetminus \{ x \}$. If $[\, y \,]_E \cap [\, y \,]_F = \{ y \}$, then $\Gamma_x \leqslant \Gamma_y$; and, since Γ_x is a maximal proper subgroup of Γ, it follows that $\Gamma_x = \Gamma_y$, which is a contradiction. Thus $[\, y \,]_F \subseteq [\, y \,]_E = [\, x \,]_E$. But then if $\gamma \in \Gamma \smallsetminus \Gamma_x$, then $[\, y \,]_F \subseteq \gamma \cdot [\, x \,]_E \cap [\, x \,]_E = \emptyset$, which is a contradiction. □

There is a natural candidate for a countable Borel equivalence relation E with an E-invariant nonatomic probability measure which does not admit an E_0-extension. Suppose that $n \geq 3$ and that p is a prime. Let $\mathbb{Z}_p$ be the ring of p-adic integers and let ν be the Haar probability measure on the compact group $SL_n(\mathbb{Z}_p)$. Since $SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$, it follows that ν is the unique $SL_n(\mathbb{Z})$-invariant probability measure under the left translation action $SL_n(\mathbb{Z}) \curvearrowright SL_n(\mathbb{Z}_p)$.

Question 1.11. Does $E_{SL_n(\mathbb{Z})}^{SL_n(\mathbb{Z}_p)}$ admit an E_0-extension?

[1] This is the Feldman-Moore Theorem [4].

[2] Recall that a transitive action $\Gamma \curvearrowright C$ is *primitive* if there are no nontrivial Γ-invariant equivalence relations on C.

Remark 1.12. It is currently unknown whether $SL_n(\mathbb{Z}) \curvearrowright SL_n(\mathbb{Z}_p)$ admits an E_0-extension.

Remark 1.13. $E_{SL_n(\mathbb{Z})}^{SL_n(\mathbb{Z}_p)}$ is not a trivial E_0-extension of itself. In order to see this, let $\Gamma = SL_n(\mathbb{Z})$ and let $X = SL_n(\mathbb{Z}_p)$. Suppose that $E \subseteq E_\Gamma^X$ is a subequivalence relation with $E \cong E_0$, which is normalized by the Borel action $\Gamma \curvearrowright X$. Then, applying Feldman-Sutherland-Zimmer [5, Theorem 4.1], since E is a normal ergodic subequivalence relation of E_Γ^X, it follows that there exists a Γ-invariant Borel subset $Y \subseteq X$ with $\nu(Y) = 1$ such that every E_Γ^X-class $C \subseteq Y$ contains only finitely many E-classes. But then, by Jackson-Kechris-Louveau [8, Proposition 1.3], $E_\Gamma^X \restriction Y$ is hyperfinite, which is a contradiction.

It is also natural to ask how many E_0-extensions up to Borel bireducibility can be realized by a fixed countable Borel equivalence relation. In fact, there exist examples of Borel actions $\Gamma \curvearrowright X$ admitting uncountably many E_0-extension up to Borel bireducibility. In more detail, if Γ is any countably infinite group, let $A_\Gamma = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where each $A_\gamma = \langle\, a_\gamma \,\rangle$ is cyclic of order 2; and let $A_\Gamma \curvearrowright 2^\Gamma$ be the Borel action defined by

$$(a_\gamma \cdot x)(g) = \begin{cases} x(g), & \text{if } g \neq \gamma; \\ 1 - x(g) & \text{if } g = \gamma. \end{cases}$$

Thus $E_0 \rtimes E_\Gamma$ is the orbit equivalence relation of the Borel action of the semidirect product $W_\Gamma = A_\Gamma \rtimes \Gamma$ on 2^Γ. Finally, let

$$\mathrm{Fr}_{W_\Gamma}(2^\Gamma) = \{\, x \in 2^\Gamma \mid g \cdot x \neq x \text{ for all } g \in W_\Gamma \smallsetminus 1 \,\}$$

be the free part of the action $W_\Gamma \curvearrowright 2^\Gamma$.

Theorem 1.14. *With the above notation, if Γ is the free group on 3 generators and $X = \mathrm{Fr}_{W_\Gamma}(2^\Gamma)$, then $\Gamma \curvearrowright X$ admits $2^{\aleph_0}$ many E_0-extension up to Borel bireducibility.*

This paper is organized as follows. In Section 2, we will recall some basic notions and results concerning countable Borel equivalence relations and ergodic theory. In Section 3, we will present the proof of Theorem 1.2. In Section 4, we will discuss the superrigidity theory that will be used in the proofs of Theorems 1.3 and 1.14.

In Section 5, we will present the proof of Theorem 1.14. The final three sections will be devoted to the proof of Theorem 1.3.

2. Preliminaries

In this section, we will recall some basic notions and results concerning countable Borel equivalence relations and ergodic theory.

2.1. **Countable Borel equivalence relations.** A detailed development of the general theory of countable Borel equivalence relations can be found in Jackson-Kechris-Louveau [8]. Here we shall only recall some basic notions and results.

A Borel equivalence relation E on a standard Borel space X is said to be *countable* if every E-class is countable. For example, if $G \curvearrowright X$ is a Borel action of a countable group G on a standard Borel space X, then the corresponding orbit equivalence relation E_G^X is a countable Borel equivalence relation. Conversely, by a classical result of Feldman-Moore [4], if E is an arbitrary countable Borel equivalence relation on a standard Borel space X, then there exists a countable group G and a Borel action $G \curvearrowright X$ such that $E = E_G^X$. A countable Borel equivalence relation E is *aperiodic* if every E-class is infinite.

If E, F are Borel equivalence relations on the standard Borel spaces X, Y, then a Borel map $f : X \to Y$ is a *homomorphism* from E to F if for all $x, y \in X$,

$$x \, E \, y \quad \Longrightarrow \quad f(x) \, F \, f(y).$$

If f satisfies the stronger property that for all $x, y \in X$,

$$x \, E \, y \quad \Longleftrightarrow \quad f(x) \, F \, f(y),$$

then f is said to be a *Borel reduction* and we write $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, then we write $E \sim_B F$ and we say that E, F are *Borel bireducible*. Finally, if both $E \leq_B F$ and $F \not\leq_B E$, then we write $E <_B F$.

If E, F are Borel equivalence relations on the standard Borel spaces X, Y, then E and F are said to be *Borel isomorphic*, written $E \cong F$, if there exists a Borel bijection $f : X \to Y$ such that for all $x, y \in X$,

$$x \, E \, y \quad \Longleftrightarrow \quad f(x) \, F \, f(y).$$

Of course, if $E \cong F$, then $E \sim_B F$. However, there are many examples of countable Borel equivalence relations E, F with $E \sim_B F$ such that $E \ncong F$.

A countable Borel equivalence relation E is *smooth* if E is Borel reducible to the equality relation Id_Z on some (equivalently every) uncountable standard Borel space Z. The countable Borel equivalence relation E on the standard Borel space X is said to be *hyperfinite* if there exists an increasing sequence

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on X such that $E = \bigcup_{n \in \mathbb{N}} F_n$. (Here a Borel equivalence relation F is said to be *finite* if every F-class is finite.) For example, the *Vitali equivalence relation*, defined on $2^{\mathbb{N}}$ by

$$x \, E_0 \, y \quad \Longleftrightarrow \quad x(n) = y(n) \text{ for all but finitely many } n,$$

is hyperfinite. By Dougherty-Jackson-Kechris [3], if E is a nonsmooth hyperfinite equivalence relation, then $E \sim_B E_0$. Finally, a countable Borel equivalence relation E is *universal* if $F \leq_B E$ for every countable Borel equivalence relation F.

2.2. **Ergodic theory.** Let $G \curvearrowright X$ be a Borel action of a countably infinite group G on a standard Borel space X. Throughout this paper, a probability measure on X will always mean a Borel probability measure; i.e. a measure which is defined on the collection of Borel subsets of X. The probability measure μ on X is *G-invariant* if $\mu(g(A)) = \mu(A)$ for every $g \in G$ and Borel subset $A \subseteq X$. If μ is G-invariant, then the Borel action $G \curvearrowright (X, \mu)$ is said to be *ergodic* if for every G-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. It is well-known that a Borel action $G \curvearrowright (X, \mu)$ is ergodic if and only if whenever Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel map, then there exists a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $f \upharpoonright X_0$ is a constant map.

The Borel action $G \curvearrowright X$ is said to be *uniquely ergodic* if there exists a unique G-invariant probability measure μ on X. In this case, it is well-known that μ must be ergodic. (For example, see Bekka-Mayer [1, Section I.3].)

Suppose that $G \curvearrowright (X, \mu)$ is ergodic and that $H \leqslant G$ is a subgroup of finite index. Then an H-invariant Borel subset $Z \subseteq X$ is said to be an *ergodic component* for the action of $H \curvearrowright (X, \mu)$ if

- $\mu(Z) > 0$; and

- H acts ergodically on (Z, μ_Z), where μ_Z is the normalized probability measure defined on Z by $\mu_Z(A) = \mu(A)/\mu(Z)$.

It is easily checked that there exists a partition $Z_1 \sqcup \cdots \sqcup Z_d$ of X into finitely many ergodic components and that the collection of ergodic components is uniquely determined up to μ-null sets. Furthermore, if the action of G on X is uniquely ergodic, then the action of H on each ergodic component $Z \subseteq X$ is also uniquely ergodic.

Suppose that E is a countable Borel equivalence relation on the standard Borel space X and that μ is a probability measure on X. Then μ is said to be *E-invariant* if μ is G-invariant for some (equivalently every) countable group G with a Borel action on X such that $E = E_G^X$. Similarly, an E-invariant probability measure μ on X is said to be *E-ergodic* if μ is ergodic for some (equivalently every) Borel action $G \curvearrowright X$ of a countable group G such that $E = E_G^X$; and E is said to be *uniquely ergodic* if there exists a unique E-invariant probability measure on X.

Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y and that μ is an E-invariant probability measure on X. Then a Borel homomorphism $f : X \to Y$ from E to F is said to be *μ-trivial* if there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single F-class. Otherwise, f is said to be *μ-nontrivial*. E is said to be *F-ergodic with respect to μ* if every Borel homomorphism from E to F is μ-trivial.

By Dougherty-Jackson-Kechris [3], the cardinality of the set of ergodic invariant probability measures is a complete invariant for Borel isomorphism of nonsmooth aperiodic hyperfinite Borel equivalence relations. The following special case is due to Nadkarni [13].

Theorem 2.1. *If E is a uniquely ergodic, aperiodic, hyperfinite Borel equivalence relation, then E is Borel isomorphic to E_0.*

3. The proof of Theorems 1.2

In this section, we will present the proof of Theorem 1.2. Let Γ be a countable group which contains a free nonabelian subgroup. Let $G \leqslant \Gamma$ be a free nonabelian subgroup, freely generated by the elements a, b. By Karras-Solitar [9, p. 950], the set $\{ a^n b^n a^n \mid n \geq 1 \}$ freely generates a malnormal[3] subgroup F of G. It is easily seen that $H = \langle aba, a^2b^2a^2 \rangle$ is a malnormal subgroup of F, and hence H is also a malnormal subgroup of G. Furthermore, it is clear that $[G : H] = \infty$.

Let E_H be the orbit equivalence relation of the shift action $H \curvearrowright 2^H$. Then, by Dougherty-Jackson-Kechris [3, Section 1], E_H is a universal countable Borel equivalence relation, and hence it is enough to show that $E_H \leq_B E_0 \rtimes E_\Gamma$. From now on, we will identify the elements of 2^Γ, 2^H with the corresponding subsets of Γ, H. Under this identification, the shift actions E_H, E_Γ correspond to the translation actions, $S \overset{g}{\mapsto} gS$; and E_0 corresponds to the almost equality relation $=^*$ on $\mathcal{P}(\Gamma)$, defined by

$$S =^* T \quad \Longleftrightarrow \quad |S \triangle T| < \infty.$$

The proof of Theorem 1.2 will make use of the following observation.

Lemma 3.1. *If $g \in G \smallsetminus H$ and c, $d \in G$, then $|gHc \cap Hd| \leq 1$.*

Proof. First note that $gHc = gHg^{-1}z$, where $z = gc$, and that $gHg^{-1} \cap H = 1$. Hence if $x \in gHc \cap Hd$, then

$$gHc \cap Hd = gHg^{-1}z \cap Hd = (gHg^{-1} \cap H)x = \{ x \}.$$

□

Let $(Hx_n \mid n \geq 0)$ enumerate the distinct cosets Hx of H in G such that $Hx \neq H$, and let $(g_n \mid n \geq 1)$ be a list of the elements of $G \smallsetminus H$ in which each $g \in G \smallsetminus H$ occurs infinitely many times. We will inductively define elements y_n, $z_n \in G \smallsetminus H$ for $n \geq 1$ such that the following conditions are satisfied:

(i) the cosets $Hx_0, Hy_1, Hz_1, \cdots, Hy_n, Hz_n$ are distinct;

(ii) $Hx_0 \cap g_n Hy_n \neq \emptyset$.

Suppose inductively that $y_1, z_1, \cdots, y_{n-1}, z_{n-1}$ have been defined. Then Lemma 3.1 implies that

$$|Hx_0 \cap g_n(H \sqcup Hx_0 \sqcup Hy_1 \sqcup Hz_1 \cdots Hy_{n-1} \sqcup Hz_{n-1})| \leq 2n,$$

and hence we can define suitable elements y_n, z_n.

[3]Recall that a subgroup F of a group G is said to be *malnormal* if $g\,F\,g^{-1} \cap F = 1$ for all $g \in G \smallsetminus F$.

Let $f : 2^H \to 2^\Gamma$ be the Borel map defined by $S \mapsto S \sqcup Hx_0 \sqcup \bigsqcup_{n \geq 1} Sz_n$. We claim that f is a Borel reduction from E_H to $E_0 \rtimes E_\Gamma$. Clearly if S, $T \in 2^H$ and $h \in H$ satisfy $hS = T$, then $hf(S) = f(T)$. Conversely, suppose that S, $T \in 2^H$ and $g \in \Gamma$ satisfy $gf(S) =^* f(T)$. If $g \in \Gamma \smallsetminus G$, then $Hx_0 \subseteq f(T)$ and $Hx_0 \cap gf(S) \subseteq G \cap gf(S) = \emptyset$, which is a contradiction. Next suppose that $g \in G \smallsetminus H$ and let $n \geq 1$ be any of the infinitely many n such that $g = g_n$. Then there exists $c_n \in Hx_0 \cap g_n Hy_n$ and clearly $c_n \notin g_n f(S)$. Since $Hx_0 \subseteq f(T)$, we again reach a contradiction. Hence $g \in H$. Clearly if $gS \neq T$, then $gf(S) \neq^* f(T)$; and so we obtain that $gS = T$. This completes the proof of Theorem 1.2.

4. Some Superrigidity Results

In this section, we will discuss the superrigidity theory that will be used in the proofs of Theorems 1.3 and 1.14. Until further notice, we will fix a Borel action $G \curvearrowright (X, \mu)$ of a countable group G on a standard Borel space X with an invariant probability measure μ.

Definition 4.1. If H is a countable group, then a Borel function $\alpha : G \times X \to H$ is called a *cocycle* if for all g, $h \in G$ and $x \in X$,

$$\alpha(hg, x) = \alpha(h, g \cdot x)\, \alpha(g, x).$$

Cocycles typically arise in the following manner. Suppose that Y is a standard Borel H-space and that H acts *freely* on Y; i.e., that $h \cdot y \neq y$ for all $y \in Y$ and $1 \neq h \in H$. If $f : X \to Y$ is a Borel homomorphism from E_G^X to E_H^Y, then we can define a corresponding Borel cocycle $\alpha : G \times X \to H$ by

$$\alpha(g, x) = \text{ the \textit{unique} element } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

Suppose now that $b : X \to H$ is a Borel map and that $f' : X \to Y$ is the "adjusted map" defined by $f'(x) = b(x) \cdot f(x)$. Then f' is also a Borel homomorphism from E_G^X to E_H^Y and the corresponding cocycle $\beta : G \times X \to H$ satisfies

$$\beta(g, x) = b(g \cdot x)\alpha(g, x)b(x)^{-1}$$

for all $g \in G$ and $x \in X$. This observation motivates the following definition.

Definition 4.2. If H is a countable group, then the cocycles α, $\beta : G \times X \to H$ are *equivalent* if there exist a Borel function $b : X \to H$ and a G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that

$$\beta(g, x) = b(g \cdot x)\alpha(g, x)b(x)^{-1}$$

for all $g \in G$ and $x \in X_0$.

Cocycle superrigidity theorems state that with suitable hypotheses on G, $(X \mu)$ and H, every Borel cocycle $\alpha : G \times X \to H$ is equivalent to a group homomorphism[4] $\varphi : G \to H$. In this case, if α is the cocycle corresponding to a Borel homomorphism $f : X \to Y$ from E_G^X to E_H^Y and $f' : X \to Y$ is the "adjusted homomorphism" corresponding to φ, then

$$\varphi(g) \cdot f'(x) = f'(g \cdot x)$$

for all $g \in G$ and $x \in X_0$; i.e., the pair $(\varphi, f' \restriction X_0)$ is a *permutation group homomorphism* from (G, X_0) to (H, Y).

The proof of Theorem 1.14 will make use of of the following special case of Popa's cocycle superrigiditity theorem [16]. (In the statement of the following theorem, we are considering the shift action $G \curvearrowright (2^G, \mu)$, together with the uniform product probability measure μ on 2^G.)

The Popa Superrigidity Theorem. *If G is a countably infinite Kazhdan group and H is any countable group, then every Borel cocycle*

$$\alpha : G \times 2^G \to H$$

is equivalent to a group homomorphism from G into H.

The proof of Theorem 1.14 involves the application of the Popa Superrigidity Theorem to a suitable family of simple quasi-finite Kazhdan groups. Recall that a countably infinite group Δ is said to be *quasi-finite* if every proper subgroup of Δ is finite. The following result is implicitly contained in Ol'shanskii [14]. (For more details, see Ozawa [15].)

[4]In some superrigidity theorems, it is necesssary to restrict to a subgroup $G_0 \leqslant G$ of finite index and an ergodic component of the restricted action $G_0 \curvearrowright (X, \mu)$.

Theorem 4.3 (Ol'shanskii [14])**.** *If H is a noncyclic torsion-free hyperbolic group, then H has a family $\{\, \Delta_\alpha = H/N_\alpha \mid \alpha < 2^{\aleph_0} \,\}$ of uncountably many pairwise nonisomorphic simple quasi-finite quotient groups.*

Corollary 4.4. *There exists a family $\{\, \Delta_\alpha =\mid \alpha < 2^{\aleph_0} \,\}$ of uncountably many pairwise nonisomorphic simple quasi-finite 2-generator Kazhdan groups.*

Proof of Corollary 4.4. Let L be a noncyclic torsion-free hyperbolic Kazhdan group. (For example, we can let L be a co-compact lattice in $Sp(n,1)$ for some $n \geq 2$. See de la Harpe-Valette [6].) Let $K \leqslant L$ be a 2-generator non-cyclic subgroup. Then, by Ol'shanskii [14, Theorem 2], L has a quotient $H = L/M$ such that:

(i) H is a noncyclic torsion-free hyperbolic group; and
(ii) $\pi(K) = H$, where $\pi : L \to L/M$ is the natural surjection.

In particular, (ii) implies that H is a 2-generator group. Applying Theorem 4.3, let $\{\, \Delta_\alpha = H/N_\alpha \mid \alpha < 2^{\aleph_0} \,\}$ be a family of uncountably many pairwise nonisomorphic simple quasi-finite quotient groups. Let $\alpha < 2^{\aleph_0}$. Since Δ_α is a quotient of H, it follows that Δ_α is a 2-generator group; and since Δ_α is a quotient of L, it follows that Δ_α is a Kazhdan group. □

Remark 4.5. For later use, suppose that $\alpha \neq \beta < 2^{\aleph_0}$ and that $\varphi : \Delta_\alpha \to \Delta_\beta$ is a group homomorphism. Since $\Delta_\alpha \not\cong \Delta_\beta$ and every proper subgroup of Δ_β is finite, it follows that φ is not an embedding; and since Δ_α is simple, it follows that φ is the trivial homomorphism such that $\varphi(g) = 1$ for all $g \in \Delta_\alpha$.

For each $\alpha < 2^{\aleph_0}$, let E_{Δ_α} be the orbit equivalence relation of the shift action $\Delta_\alpha \curvearrowright 2^{\Delta_\alpha}$ and let μ_α denote the uniform product probability measure on 2^{Δ_α}. The proof of Theorem 1.14 will make use of the following strong Borel incomparability result.

Theorem 4.6. *If $\alpha \neq \beta$, then E_{Δ_α} is E_{Δ_β}-ergodic with respect to μ_α.*

Proof. Suppose not and let $f : 2^{\Delta_\alpha} \to 2^{\Delta_\beta}$ be a μ_α-nontrivial Borel homomorphism from E_{Δ_α} to E_{Δ_β}. Let

$$\mathrm{Fr}_{\Delta_\beta}(2^{\Delta_\beta}) = \{\, x \in 2^{\Delta_\beta} \mid g \cdot x \neq x \text{ for all } g \in \Delta_\beta \smallsetminus 1 \,\}$$

be the free part of the action $\Delta_\beta \curvearrowright 2^{\Delta_\beta}$. Applying Thomas [19, Lemma 7.8], since Δ_β is a simple quasi-finite group, it follows that E_{Δ_β} is Borel bireducible with $E_{\Delta_\beta} \restriction \mathrm{Fr}_{\Delta_\beta}(2^{\Delta_\beta})$. Hence we can suppose that $f(x) \in \mathrm{Fr}_{\Delta_\beta}(2^{\Delta_\beta})$ for all $x \in 2^{\Delta_\alpha}$; and we can define a Borel cocycle $\alpha : \Delta_\alpha \times 2^{\Delta_\alpha} \to \Delta_\beta$ by

$$\alpha(g, x) = \text{ the unique } h \in \Delta_\beta \text{ such that } h \cdot f(x) = f(g \cdot x).$$

By the Popa Superrigidity Theorem, α is equivalent to a group homomorphism $\varphi : \Delta_\alpha \to \Delta_\beta$; and, by Remark 4.5, φ is necessarily the trivial homomorphism such that $\varphi(g) = 1$ for all $g \in \Delta_\alpha$. Thus there exist a Borel map $b : 2^{\Delta_\alpha} \to \Delta_\beta$ and a Δ_α-invariant Borel subset $X_0 \subseteq 2^{\Delta_\alpha}$ with $\mu_\alpha(X_0) = 1$ such that

$$b(g \cdot x)\alpha(g, x)b(x)^{-1} = 1$$

for all $g \in \Delta_\alpha$ and $x \in X_0$. Let $f' : 2^{\Delta_\alpha} \to \Delta_\beta$ be the Borel map defined by $f'(x) = b(x) \cdot f(x)$. Then $f' \restriction X_0$ is Δ_α-invariant; and hence, by ergodicity, there exists a Borel subset $X_1 \subseteq X_0$ with $\mu_\alpha(X_1) = 1$ such that such that $f' \restriction X_1$ is a constant map. But this means that f maps X_1 into a single E_{Δ_β}-class, which contradicts the assumption that f is μ_α-nontrivial. □

The proof of Theorem 1.3 will make use of the following result, which is an easy consequence of the Ioana cocycle superrigidity theorem [7] for profinite actions of Kahzdan groups. Let $n \geq 3$, let $V = \prod_{p \in \mathbb{P}} V(n, p)$, and let $\Gamma = SL_n(\mathbb{Z})$. Then we can define a Γ-invariant ergodic probability measure ν for the Borel action $\Gamma \curvearrowright V$ as follows.[5] For each $p \in \mathbb{P}$, let ν_p be the probability measure on $V(n, p)$ defined by

$$\nu_p(v) = \begin{cases} 1/(p^n - 1), & \text{if } v \neq 0; \\ 0 & \text{if } v = 0. \end{cases}$$

Then the corresponding product measure $\nu = \prod_{p \in \mathbb{P}} \nu_p$ on V is Γ-invariant and ergodic.

Theorem 4.7. *Let $n \geq 3$ and let $\Gamma = SL_n(\mathbb{Z})$. If H is any countable group and*

$$\alpha : \Gamma \times V \to H$$

[5] In Section 6, we will classify all of the Γ-invariant ergodic probability measure on V.

is a Borel cocycle, then there exists a subgroup $\Delta \leqslant \Gamma$ *with* $[\,\Gamma : \Delta\,] < \infty$ *and an ergodic component* X_0 *for the action of* $\Delta \curvearrowright (\,V, \nu\,)$ *such that* $\alpha \restriction (\,\Delta \times X_0\,)$ *is equivalent to either:*

(a) *an embedding* $\varphi : \Delta \to H$*; or else*

(b) *the trivial homomorphism* $\varphi : \Delta \to H$ *which takes constant value* 1*.*

Remark 4.8. The proof of Theorem 4.7 is virtually identical to that of Thomas [20, Theorem 4.3]. There is a slight complication when n is even and $Z(SL_n(\mathbb{Z}))$ is a nontrivial finite normal subgroup of Γ. However, this is easily dealt with by first passing to a subgroup $\Gamma_0 \leqslant \Gamma$ of finite index such that $\Gamma_0 \cap Z(SL_n(\mathbb{Z})) = 1$.

The proof of Theorem 1.3 will also make use of the following non-embeddability theorem, which is an easy consequence of the Margulis superrigidity theorems [11].

Theorem 4.9. *Let* $n \geq 3$ *and let* $\Gamma = SL_n(\mathbb{Z})$*. If* $\Delta \leq \Gamma$ *is a subgroup with* $[\,\Gamma : \Delta\,] < \infty$ *and* $1 \leq m < n$*, then* Δ *does not embed into* $GL_m(\mathbb{C})$*.*

5. The proof of Theorems 1.14

In this section, we will present the proof of Theorem 1.14. First we will recall the notation that we will use throughout this section. Let Γ be a countably infinite group, let E_Γ be the orbit equivalence relation of the shift action $\Gamma \curvearrowright 2^\Gamma$ and let μ be the uniform product probability measure on 2^Γ. Throughout, the Vitali equivalence relation on 2^Γ will be denoted by E_0. Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where each $A_\gamma = \langle\, a_\gamma \,\rangle$ is cyclic of order 2; and let $A \curvearrowright 2^\Gamma$ be the Borel action defined by

$$(a_\gamma \cdot x)(g) = \begin{cases} x(g), & \text{if } g \neq \gamma; \\ 1 - x(g) & \text{if } g = \gamma. \end{cases}$$

Thus E_0 is the orbit equivalence relation of the action $A \curvearrowright 2^\Gamma$; and $E_0 \rtimes E_\Gamma$ is the orbit equivalence relation of the action of $W_\Gamma = A \rtimes \Gamma$ on 2^Γ. Finally, for each $H \in \{\,\Gamma, W_\Gamma\,\}$, let

$$\mathrm{Fr}_H(2^\Gamma) = \{\, x \in 2^\Gamma \mid h \cdot x \neq x \text{ for all } h \in H \smallsetminus 1 \,\}$$

be the free part of the action $H \curvearrowright 2^\Gamma$. Then clearly $\mathrm{Fr}_H(2^\Gamma)$ is an H-invariant Borel subset of 2^Γ. In this section, we will continue to identify each $x \in 2^\Gamma$ with the corresponding subset $S \subseteq \Gamma$ and to identify E_0 with the almost equality relation $=^*$ on $\mathcal{P}(\Gamma)$.

Lemma 5.1. $\mu(\operatorname{Fr}_{W_\Gamma}(2^\Gamma)) = 1$.

Proof. It is enough to show that

$$\mu(\{ x \in 2^\Gamma \mid g \cdot x = x \}) = 0$$

for all $1 \neq g \in W_\Gamma$. Let $g = a\gamma$, where $a \in A$ and $\gamma \in \Gamma$. Since A acts freely on 2^Γ, we can suppose that $\gamma \neq 1$. Note that if $S \in 2^\Gamma$ and $g \cdot S = S$, then $\gamma \cdot S =^* S$. Let $C = \langle \gamma \rangle$ be the cyclic subgroup generated by γ.

First suppose that C is finite and let T be a set of right coset representatives of C in Γ. Then T is infinite and $\gamma \cdot S =^* S$ if and only if for all but finitely many $t \in T$, we have that either $S \cap Ct = Ct$ or $S \cap Ct = \emptyset$. There exists a fixed $0 < p < 1$ such that for all $t \in T$,

$$\mu(\{ S \in 2^\Gamma \mid S \cap Ct \neq Ct \text{ and } S \cap Ct \neq \emptyset \}) = p.$$

By the second Borel-Cantelli Lemma, since these events are independent, it follows that for μ-a.e. $S \in 2^\Gamma$, there exist infinitely many $t \in T$ such that $S \cap Ct \neq Ct$ and $S \cap Ct \neq \emptyset$.

Next suppose that C is an infinite cyclic group. Then if $\gamma \cdot S =^* S$, we must have that either:

(i) $S \cap C$ is finite; or
(ii) there exists $\ell \in \mathbb{Z}$ such that $\{ \gamma^n \mid n \leq \ell \} \subseteq S \cap C$; or
(iii) there exists $\ell \in \mathbb{Z}$ such that $\{ \gamma^n \mid n \geq \ell \} \subseteq S \cap C$.

Clearly the set of $S \in 2^\Gamma$ satisfying conditions (i), (ii) or (iii) forms a μ-null set. □

From now on, let $\Gamma = \langle a, b, c \rangle$ be the free group on the generators a, b, c and let $F = \langle a, b \rangle$ be the free group on the generators a, b. Let $\pi : \Gamma \to F$ be the surjective homomorphism such that $\pi(a) = a$, $\pi(b) = b$ and $\pi(c) = 1$; and let $N = \ker \pi$. Then clearly F is a set of coset representatives of N in Γ. Let $X_F \subseteq 2^\Gamma$ be the Borel subset consisting of the subsets $S \subseteq \Gamma$ such that there exists a (possibly empty) subset $I \subseteq F$ with $S = \bigsqcup_{t \in I} Nt$. For each $S \in 2^\Gamma$, let $[S]_{E_0}$ be the corresponding E_0-class.

Claim 5.2. *If $S \in X_F$, then $[\,S\,]_{E_0} \smallsetminus \mathrm{Fr}_\Gamma(2^\Gamma) = \{\,S\,\}$.*

Proof of Claim 5.2. Suppose that $S = \bigsqcup_{t \in I} Nt \in X_F$. Then clearly $g \cdot S = S$ for each $g \in N$ and so $S \notin \mathrm{Fr}_\Gamma(2^\Gamma)$. Let $S' \in [\,S\,]_{E_0} \smallsetminus \{\,S\,\}$. Then at least one of the following possibilities must hold:

(i) $\{\, t \in T \mid 0 < |S' \cap Nt| < \infty \,\}$ is a nonempty finite subset of $T \smallsetminus I$;
(ii) $\{\, t \in T \mid 0 < |Nt \smallsetminus S'| < \infty \,\}$ is a nonempty finite subset of T.

Suppose that $g \in \Gamma$ satisfies $g \cdot S' = S'$. Since N is a normal subgroup of Γ, it follows that g permutes the cosets $\{\, Nt \mid t \in T \,\}$; and since at one of (i) or (ii) holds, it follows that there exists a nonempty finite subset $A \subseteq \Gamma$ such that $g \cdot A = A$, and this implies that $g = 1$. □

Let $Y = \bigcup \{\, \mathrm{Fr}_\Gamma(2^\Gamma) \cap [\,S\,]_{E_0} \mid S \in X_F \,\}$. Note that Claim 5.2 implies that if $S \in X_F$ and $1 \neq a \in A$, then $a \cdot S \in Y$.

Claim 5.3. *If $1 \neq a \in A$, then the map $S \mapsto a \cdot S$ is a Borel reduction from $E_\Gamma \restriction X_F$ to $(E_0 \rtimes E_\Gamma) \restriction Y$.*

Proof of Claim 5.3. Suppose that $S_1,\ S_2 \in X_F$. If there exists $\gamma \in \Gamma$ such that $\gamma \cdot S_1 = S_2$, then $\gamma \cdot [\,S_1\,]_{E_0} = [\,S_2\,]_{E_0}$ and it follows that there exists $g \in W_\Gamma$ such that $g \cdot (a \cdot S_1) = a \cdot S_2$. Conversely, suppose that there exists $g \in W_\Gamma$ such that $g \cdot (a \cdot S_1) = a \cdot S_2$. Let $g = \gamma b$, where $\gamma \in \Gamma$ and $b \in A$. Then $\gamma \cdot [\,S_1\,]_{E_0} = [\,S_2\,]_{E_0}$, and Claim 5.2 implies that $\gamma \cdot S_1 = S_2$. □

Note that the above argument also yields the following result.

Claim 5.4. *Let $\sigma : Y \to X_F$ be the Borel map defined by*

$$\sigma(y) = \text{ the unique } z \in X_F \text{ such that } z \, E_0 \, y.$$

Then σ is a Borel reduction from $(E_0 \rtimes (E_\Gamma) \restriction Y$ to $E_\Gamma \restriction X_F$.

Applying Corollary 4.4, let $\{\, \Delta_\alpha =\mid \alpha < 2^{\aleph_0} \,\}$ be a family of uncountably many pairwise nonisomorphic simple quasi-finite 2-generator Kazhdan groups. For each $\alpha < 2^{\aleph_0}$, let $\pi_\alpha : F \to \Delta_\alpha$ be a surjective homomorphism and let

$$f_\alpha : 2^{\Delta_\alpha} \to X_F$$

be the Borel injection defined by

$$S \mapsto \bigsqcup_{t \in \pi_\alpha^{-1}(S)} Nt. \tag{5.3}$$

Let $X_{\Delta_\alpha} = f_\alpha(2^{\Delta_\alpha})$. Then X_{Δ_α} is a Γ-invariant Borel subset of X_F and f_α is a Borel isomorphism from E_{Δ_α} to $E_\Gamma \restriction X_{\Delta_\alpha}$. Let

$$Y_\alpha = \bigcup \{ \operatorname{Fr}_\Gamma(2^\Gamma) \cap [\, S \,]_{E_0} \mid S \in X_{\Delta_\alpha} \}$$

and let $Z_\alpha = \operatorname{Fr}_{W_\Gamma}(2^\Gamma) \sqcup Y_\alpha$. Then clearly Y_α, Z_α are Γ-invariant Borel subsets of $\operatorname{Fr}_\Gamma(2^\Gamma)$ and Γ permutes the $(E_0 \restriction Z_\alpha)$-classes.

Claim 5.6. *$E_0 \restriction Z_\alpha$ is Borel isomorphic to E_0.*

Proof of Claim 5.6. Applying Claim 5.2, we see that the following conditions hold:

(i) $E_0 \restriction Y_\alpha$ is aperiodic; and

(ii) $E_0 \restriction Y_\alpha$ is smooth.

Since $E_0 \restriction \operatorname{Fr}_{W_\Gamma}(2^\Gamma)$ is clearly aperiodic, it follows that $E_0 \restriction Z_\alpha$ is aperiodic. Also condition (ii) implies that if ν is an $(E_0 \restriction Z_\alpha)$-invariant probability measure on Z_α, then $\nu(\, Y_\alpha \,) = 0$. (For example, see Dougherty-Jackson-Kechris [3].) Since $E_0 \restriction \operatorname{Fr}_{W_\Gamma}(2^\Gamma)$ is uniquely ergodic, it follows that $E_0 \restriction Z_\alpha$ is also uniquely ergodic. Hence the result follows from Theorem 2.1. □

Claim 5.7. *$\Gamma \curvearrowright Z_\alpha$ is Borel isomorphic to $\Gamma \curvearrowright \operatorname{Fr}_{W_\Gamma}(2^\Gamma)$.*

Proof of Claim 5.7. It is enough to show that $E_\Gamma \restriction Y_\alpha$ is smooth. To see this, first let $\{\, A_n \mid n \in \mathbb{N} \,\}$ be an enumeration of the finite nonempty subsets of Γ. Now suppose that $C \subseteq Y_\alpha$ is a Γ-orbit; and for each $S' \in C$, let $S \in X_{\Delta_\alpha}$ be the unique element such that $S' \mathrel{E_0} S$. Then $S = \bigsqcup_{t \in I_S} Nt$ for some subset $I_S \subseteq T$; and

$$\mathcal{F}_{S'} = \bigcup \{\, S' \cap Nt \mid 0 < |S' \cap Nt| < \infty \,\} \cup \bigcup \{\, Nt \smallsetminus S' \mid 0 < |Nt \smallsetminus S'| < \infty \,\}$$

is a nonempty finite subset of Γ. Furthermore, it is clear that if $\gamma \in \Gamma$, then $\mathcal{F}_{\gamma \cdot S'} = \gamma \cdot \mathcal{F}_{S'}$. It follows that the map

$$C \to \{\, A_n \mid n \in \mathbb{N} \,\}$$

$$S' \mapsto \mathcal{F}_{S'}$$

is injective; and so we can make a Borel selection of an element $S'_C \in C$ by letting S'_C be the unique $S' \in C$ such that $\mathcal{F}_{S'} = A_{n(C)}$, where

$$n(C) = \min\{\, n \in \mathbb{N} \mid (\,\exists S' \in C\,)\, \mathcal{F}_{S'} = A_n \,\}.$$

□

Combining Claims 5.6 and 5.7, we see that $(E_0 \restriction Z_\alpha) \rtimes (E_\Gamma \restriction Z_\alpha)$ is Borel isomorphic to an E_0-extension of $\Gamma \curvearrowright \mathrm{Fr}_{W_\Gamma}(2^\Gamma)$. Hence, in order to complete the proof of Theorem 1.14, it is enough to prove the following result.

Theorem 5.8. *If $\alpha \neq \beta$, then $(E_0 \restriction Z_\alpha) \rtimes (E_\Gamma \restriction Z_\alpha)$ is not Borel reducible to $(E_0 \restriction Z_\beta) \rtimes (E_\Gamma \restriction Z_\beta)$.*

Suppose that $f : Z_\alpha \to Z_\beta$ is a Borel reduction from $(E_0 \restriction Z_\alpha) \rtimes (E_\Gamma \restriction Z_\alpha)$ to $(E_0 \restriction Z_\beta) \rtimes (E_\Gamma \restriction Z_\beta)$. Let $f_\alpha : 2^{\Delta_\alpha} \to X_{\Delta_\alpha}$ be the Borel isomorphism from E_{Δ_α} to $E_\Gamma \restriction X_{\Delta_\alpha}$ given by (5.3) and let $1 \neq a \in A$ be a fixed nonidentity element. Let $\varphi : 2^{\Delta_\alpha} \to Z_\beta$ be the Borel map defined by

$$S \mapsto f(\,a \cdot f_\alpha(S)\,).$$

Applying Claim 5.3, we see that $S \mapsto a \cdot f_\alpha(S)$ is a Borel reduction from E_{Δ_α} to $(E_0 \restriction Z_\alpha) \rtimes (E_\Gamma \restriction Z_\alpha)$; and it follows that φ is a Borel reduction from E_{Δ_α} to $(E_0 \restriction Z_\beta) \rtimes (E_\Gamma \restriction Z_\beta)$. Since $\Delta_\alpha \curvearrowright (\,2^{\Delta_\alpha}, \mu_\alpha\,)$ is ergodic, either:

- $\varphi(x) \in \mathrm{Fr}_{W_\Gamma}(2^\Gamma)$ for μ_α-a.e. $x \in 2^{\Delta_\alpha}$; or
- $\varphi(x) \in Y_\beta$ for μ_α-a.e. $x \in 2^{\Delta_\alpha}$.

Case 1: Suppose that $\varphi(x) \in \mathrm{Fr}_{W_\Gamma}(2^\Gamma)$ for μ_α-a.e. $x \in 2^{\Delta_\alpha}$. Then, after deleting a μ_α-null subset of 2^{Δ_α}, we can define a Borel cocycle $\alpha : \Delta_\alpha \times 2^{\Delta_\alpha} \to W_\Gamma$ by

$$\alpha(\gamma, x) = \text{ the unique } g \in W_\Gamma \text{ such that } g \cdot \varphi(x) = \varphi(\gamma \cdot x).$$

Applying the Popa Superrigidity Theorem, it follows that α is equivalent to a group homomorphism $\psi : \Delta_\alpha \to W_\Gamma$. Hence, after adjusting φ, we can suppose that for μ_α-a.e. $x \in 2^{\Delta_\alpha}$, for all $\gamma \in \Delta_\alpha$,

$$\psi(\gamma) \cdot \varphi(x) = \varphi(\gamma \cdot x).$$

It is clear that the simple quasi-finite group Δ_α does not embed into $W_\Gamma = A \rtimes \Gamma$; and it follows that ψ is the trivial homomorphism such that $\psi(\gamma) = 1$ for all

$\gamma \in \Delta_\alpha$. Thus φ is μ_α-a.e. Δ_α-invariant; and by ergodicity, this implies that φ is μ_α-a.e. constant, which is a contradiction.

Case 2: Suppose that $\varphi(x) \in Y_\beta$ for μ_α-a.e. $x \in 2^{\Delta_\alpha}$. Applying Claim 5.2, let $\sigma : Y_\beta \to X_{\Delta_\beta}$ be the Borel map defined by

$$\sigma(y) = \text{ the unique } z \in X_{\Delta_\beta} \text{ such that } z \, E_0 \, y.$$

Then, by Claim 5.4, σ is a Borel reduction from $(E_0 \rtimes (E_\Gamma) \restriction Y_\beta$ to $E_\Gamma \restriction X_{\Delta_\beta}$. It follows that the map $x \mapsto (f_\beta^{-1} \circ \sigma \circ \varphi)(x)$ is μ_α-a.e. a Borel reduction from E_α to E_β, which contradicts Theorem 4.6.

It should be pointed out that the current version of Theorem 1.14 is situated very much in the Borel setting, rather than in the measurable setting in which sets and functions are identified if they agree on a conull set. This suggests the following problem.

Notation 5.9. For each countable Borel equivalence relation E such that $E \cong E_0$, let μ_E be the unique E-invariant probability measure.

Question 5.10. Does there exist a countable Borel equivalence relation F on a standard Borel space X with uncountably many E_0-extensions $\{ E_\alpha \rtimes F \mid \alpha < 2^{\aleph_0} \}$ such that whenever $Y \subseteq X$ is a Borel subset with $\mu_{E_\alpha}(Y) = 1$ and $\beta \neq \alpha$, then $(E_\alpha \rtimes F) \restriction Y$ is not Borel reducible to $E_\beta \rtimes F$?

6. Ergodic probability measures

The remaining sections of this paper will be devoted to the proof of Theorem 1.3. From now on, we will let $\Gamma = SL_n(\mathbb{Z})$. Recall that E_Γ^V is the orbit equivalence relation corresponding to the Borel action

$$\Gamma = SL_n(\mathbb{Z}) \curvearrowright V = \prod_{p \in \mathbb{P}} V(n,p),$$

and that E_0^V is the countable Borel equivalence relation on V defined by

$$(v_p) \, E_0^V \, (w_p) \quad \Longleftrightarrow \quad v_p = w_p \text{ for all but finitely many } p \in \mathbb{P}.$$

In order to simply notation, for the remainder of this paper, we will write E, E^* instead of E_Γ^V, $E_0^V \rtimes E_\Gamma^V$ respectively. For each $p \in \mathbb{P}$, let $T_p = \{ t_v \mid v \in V(n,p) \}$, where

$$t_v(w) = v + w, \qquad w \in V(n,p).$$

Let $T = \bigoplus_{p \in \mathbb{P}} T_p$ and let $G = T \rtimes \Gamma$. Then clearly E^* is the orbit equivalence relation corresponding to the Borel action $G \curvearrowright V$.

In this section, in preparation for the proof of Theorem 1.3, we will classify the Γ-invariant and G-invariant ergodic probability measures on V. First note that Γ is a dense subgroup of the compact group $K_0 = \prod_{p \in \mathbb{P}} SL(n,p)$. It follows that G is a dense subgroup of the compact group $K = \prod_{p \in \mathbb{P}} [\, T_p \rtimes SL(n,p)\,]$. Since K acts transitively on V, it follows that the action $G \curvearrowright V$ is uniquely ergodic; and, since G clearly preserves the product μ of the uniform probability measures on the $V(n,p)$, it follows that μ is the unique G-invariant probability measure on V.

The situation for the action $\Gamma \curvearrowright V$ is a little more complex. First note that the orbits of K_0 on V are precisely $\{\, V_S \mid S \subseteq \mathbb{P} \,\}$, where

$$V_S = \{\, (v_p) \in V \mid v_p \neq 0 \iff p \in S \,\}.$$

It follows that each action $\Gamma \curvearrowright V_S$ is uniquely ergodic. Let

$$V_S(n,p) = \begin{cases} V(n,p) \smallsetminus \{\, 0 \,\}, & \text{if } p \in S; \\ \{\, 0 \,\}, & \text{otherwise.} \end{cases}$$

Then $V_S = \prod_{p \in \mathbb{P}} V_S(n,p)$ and the unique Γ-invariant probability measure on V_S is the product μ_S of the uniform probability measures on the sets $V_S(n,p)$.

Lemma 6.1. *The Γ-invariant ergodic probability measures on V are $\{\, \mu_S \mid S \subseteq \mathbb{P} \,\}$.*

Proof. Suppose that ν is a Γ-invariant ergodic probability measures on V. Since Γ is a dense subgroup of K_0, it follows that ν is also K_0-invariant. Since K_0 is compact, every K_0-invariant ergodic probability measure on V is supported on a K_0-orbit. (For example, see Bekka-Mayer [1, Corollary 2.21].) It follows that $\nu = \mu_S$ for some $S \subseteq \mathbb{P}$. □

Finally, we will also need to know the Δ-invariant ergodic probability measures on V for subgroups $\Delta \leqslant \Gamma$ of finite index. Recall that if ν is a probability measure on a topological space X, then the *support* of ν, denoted by $\mathrm{supp}(\nu)$, is the smallest closed subset $C \subseteq X$ such that $\nu(X \smallsetminus C) = 0$.

Lemma 6.2. *Suppose that $\Delta \leqslant \Gamma$ is a subgroup with $[\, \Gamma : \Delta \,] < \infty$ and that ν is an ergodic probability measure for the action $\Delta \curvearrowright V$. Then there exists a subset*

$S \subseteq \mathbb{P}$ such that $\operatorname{supp}(\nu)$ *is an ergodic component of* $\Delta \curvearrowright (V, \mu_S)$ *and* ν *is the normalized probability measure on* $\operatorname{supp}(\nu)$ *corresponding to* μ_S*.*

Proof. Let H be the closure of Δ in $K_0 = \prod_{p \in \mathbb{P}} SL(n,p)$. Then ν is also H-invariant and hence is supported on an H-orbit; say, $H \cdot v$. Furthermore, the action of $H \curvearrowright H \cdot v$ is uniquely ergodic. Let $S \subseteq \mathbb{P}$ be the subset such that $K_0 \cdot v = V_S$. Then the normalized probability measure on $H \cdot v$ corresponding to μ_S is H-invariant and hence is equal to ν. □

7. The proof of Theorem 1.3

Let $\nu = \mu_{\mathbb{P}}$ be the Γ-invariant probability measure which concentrates on

$$V_{\mathbb{P}} = \{ (v_p) \in V \mid v_p \neq 0 \text{ for all } p \in \mathbb{P} \},$$

and let $\operatorname{Fr}_G(V) = \{ v \in V \mid g \cdot v \neq v \text{ for all } g \in G \smallsetminus 1 \}$. Equivalently, $v \in \operatorname{Fr}_G(V)$ if and only if for all $1 \neq \gamma \in \Gamma$, v and $\gamma \cdot v$ are not E_0^V-equivalent. The proof of the following technical result will be delayed until Section 8.

Lemma 7.1. *If $f : V \to V$ is a ν-nontrivial Borel homomorphism from E to E^*, then* $\nu(\{ v \in V \mid f(v) \in \operatorname{Fr}_G(V) \}) = 1$.

In this section, we will present the proof of Theorem 1.3, modulo Lemma 7.1. First suppose that that $f : V \to V$ is a Borel reduction from E^* to E. Then f is a ν-nontrivial Borel homomorphism from E to E^*. Hence, by Lemma 7.1, there exists a Γ-invariant Borel subset $X \subseteq V$ with $\nu(X) = 1$ such that $f(v) \in \operatorname{Fr}_G(V)$ for all $v \in X$; and we can define a Borel cocycle $\alpha : \Gamma \times X \to \Gamma$ by

$$\alpha(g,v) = h, \text{ where } h \in \Gamma \text{ is the unique element such that } h \cdot f(v) = f(g \cdot v).$$

By Theorem 4.7, there exists a subgroup $\Delta \leqslant \Gamma$ with $[\Gamma : \Delta] < \infty$ and an ergodic component $X_0 \subseteq X$ for the action $\Delta \curvearrowright X$ such that $\alpha \restriction (\Delta \times X_0)$ is equivalent to either an embedding $\varphi : \Delta \to \Gamma$ or else the trivial homomorphism $\varphi : \Delta \to \Gamma$ which takes constant value 1. Thus, after deleting a null subset of X_0 if necessary, we can suppose that there exists a Borel map $b : X_0 \to \Gamma$ such that

$$\varphi(g) = b(g \cdot v)\alpha(g,v)b(v)^{-1}$$

for all $v \in X_0$ and $g \in \Delta$. Let $f' : X_0 \to V$ be the Borel map defined by $f'(v) = b(v) \cdot f(v)$. Then

$$f'(g \cdot v) = \varphi(g) \cdot f'(v)$$

for all $v \in X_0$ and $g \in \Delta$. If φ is the trivial homomorphism, then $f' : X_0 \to V$ is Δ-invariant; and hence, since Δ acts ergodically on X_0, it follows that f' is ν-a.e. constant. But this means that f maps ν-a.e. $v \in X_0$ into a single E-class, which is a contradiction. Thus we can suppose that φ is an embedding; and it follows that $[\,\Gamma : \varphi(\Delta)\,] < \infty$. In order to simplify notation, we will suppose that $f' = f$, so that

$$f(g \cdot v) = \varphi(g) \cdot f(v)$$

for all $v \in X_0$ and $g \in \Delta$. Next note that by applying Lemma 7.1 to the identity map $V \to V$, it follows that $\nu(\mathrm{Fr}_G(V)) = 1$. (Lemma 7.3 will provide a direct proof of this result.) Hence can suppose that $X_0 \subseteq X \subseteq \mathrm{Fr}_G(V)$. It follows that if $v \in X_0$, then f restricts to a bijection between the orbits $\Delta \cdot v$ and $\varphi(\Delta) \cdot f(v)$. Furthermore, if $v \in X_0$ and $1 \neq g \in \Gamma$, then v and $g \cdot v$ are not E_0^V-equivalent. Also, since $\nu(X_0) > 0$, it follows that $E_0^V \restriction X_0$ is not smooth and so we can suppose that $[v]_{E_0^V} \cap X_0$ is infinite for each $v \in X_0$. Fix some $v \in X_0$ and let $v = v_0, v_1, \cdots, v_n, \cdots$ enumerate $[v]_{E_0^V} \cap X_0$. Then the orbits $\{\, \Delta \cdot v_n \mid n \in \mathbb{N} \,\}$ are distinct; and $\varphi(\Delta) \cdot f(v_n) \subseteq \Gamma \cdot f(v)$ for all $n \in \mathbb{N}$. Since $[\,\Gamma : \varphi(\Delta)\,] < \infty$, it follows that $\Gamma \cdot f(v)$ contains only finitely many $\varphi(\Delta)$-orbits. Let $\bar{\nu}$ be the Δ-ergodic probability measure on X_0 defined by $\bar{\nu}(A) = \nu(A)/\nu(X_0)$ Let $Z = f(X_0)$ and let $m = f_* \bar{\nu}$. Then $\varphi(\Delta)$ acts ergodically on the standard probability space $(\,Z, m\,)$. By ergodicity, we can suppose that each $\varphi(\Delta)$-orbit is the image of the same number of Δ-orbits; and this number is necessarily infinite. Thus we can define a smooth aperiodic countable Borel equivalence relation $F \subseteq E_\Delta^{X_0}$ by

$$v \, F \, w \iff f(v) = f(w).$$

But, by Dougherty-Jackson-Kechris [3, Proposition 2.5], this means that $E_\Delta^{X_0}$ is compressible, which is a contradiction. (Recall that, by Nadkarni [12], if a countable Borel equivalence relation E is compressible, then E does not admit an invariant probability measure.)

Next suppose that $f : V \to V$ is a Borel reduction from E to E^*. Applying Lemma 7.1, there exists a Γ-invariant Borel subset $X \subseteq V$ with $\nu(X) = 1$ such that $f(v) \in \mathrm{Fr}_G(V)$ for all $v \in X$; and we can define a Borel cocycle $\alpha : \Gamma \times X \to G$ by

$$\alpha(g, v) = h, \text{ where } h \in G \text{ is the unique element such that } h \cdot f(v) = f(g \cdot v).$$

By Theorem 4.7, there exists a subgroup $\Delta \leqslant \Gamma$ with $[\,\Gamma : \Delta\,] < \infty$ and an ergodic component $X_0 \subseteq X$ for the action $\Delta \curvearrowright X$ such that $\alpha \restriction (\,\Delta \times X_0\,)$ is equivalent to either an embedding $\varphi : \Delta \to G$ or else the trivial homomorphism $\varphi : \Delta \to G$ which takes constant value 1. Furthermore, arguing as above, we can suppose that φ is an embedding; and, after deleting a null subset of X_0 and adjusting f if necessary, we can also suppose that

$$f(g \cdot v) = \varphi(g) \cdot f(v)$$

for all $v \in X_0$ and $g \in \Delta$. Since Γ is finitely generated and $[\,\Gamma : \Delta\,] < \infty$, it follows that Δ is also finitely generated. Thus there exists a finite subset $F \subset \mathbb{P}$ such that

$$\varphi(\Delta) \leqslant [\bigoplus_{p \in F} T_p] \rtimes \Gamma.$$

Suppose that v, $w \in X_0$ and that $g \in \Delta$ satisfies $g \cdot v = w$. Then $\varphi(g) \cdot f(v) = f(w)$. Let $\varphi(g) = th$, where $t \in \bigoplus_{p \in F} T_p$ and $h \in \Gamma$. Let $f(v) = (u_p)$ and $f(w) = (z_p)$. Then $h \cdot u_p = z_p$ for all $p \in \mathbb{P} \smallsetminus F$.

Let $f' : X_0 \to V$ be the Borel map defined by

$$f'(v) = \begin{cases} f(v)_p, & \text{if } p \in \mathbb{P} \smallsetminus F; \\ 0, & \text{if } p \in F. \end{cases}$$

Also, let $\varphi' : \Delta \to \Gamma$ be the homomorphism defined by $\varphi'(g) = h$, where $\varphi(g) = th$ for some $t \in \bigoplus_{p \in F} T_p$. Then clearly $f'(v)\, E^*\, f(v)$ and so f' is a Borel reduction from $E \restriction X_0$ to E^*. Furthermore, if $v \in X_0$ and $g \in \Delta$, then

$$f'(g \cdot v) = \varphi'(g) \cdot f'(v)$$

and so $f'(g \cdot v)\, E\, f'(v)$. Let $Z = f'(X_0)$. Then the above remarks imply that if C is an E^*-class of V, then $C \cap Z$ is contained in a single E-class. In order to simplify notation, we will suppose that $f' = f \restriction X_0$ and that $\varphi' = \varphi$.

Let $\bar{\nu}$ be the Δ-ergodic probability measure on X_0 defined by $\bar{\nu}(A) = \nu(A)/\nu(X_0)$ and let $m = f_* \bar{\nu}$. Then $\varphi(\Delta)$ acts ergodically on the standard probability space

(Z, m). By Lemma 6.2, there exists a subset $S \subseteq \mathbb{P}$ such that m is the normalized probability measure on Z corresponding to the finite measure $\mu_S \restriction Z$ and it is clear that S cannot be a finite subset of $\mathbb{P}$. Thus, in order to reach a contradiction, it is enough to prove the following result.

Proposition 7.2. *Suppose that $S \subseteq \mathbb{P}$ is an infinite subset and that $Z \subseteq V$ is a Borel subset such that if C is an E^*-class of V, then $C \cap Z$ is contained in a single E-class. Then $\mu_S(Z) = 0$.*

First we need to prove the following result.

Lemma 7.3. *If $S \subseteq \mathbb{P}$ is an infinite subset, then $\mu_S(V_S \cap \mathrm{Fr}_G(V)) = 1$.*

Proof. For each $1 \neq g \in \Gamma$, let W_S^g be the subset of those $v \in V_S$ such that v and $g \cdot v$ are not E_0-equivalent. Then it is enough to show that each $\mu_S(W_S^g) = 1$. To see this, for each $p \in S$, let $g_p \in SL(n,p)$ be the corresponding element. Then $g_p \neq 1$ for all but finitely many $p \in S$. Let $T \subseteq \{ p \in S \mid g_p \neq 1 \}$ be an infinite subset such that $\sum_{p \in T} 1/p < \infty$. For each $p \in T$, let $\mathrm{Fix}(g_p) \leqslant V(n,p)$ be the subspace of vectors fixed by g_p and let $E_p^g = \{ v \in V \mid v_p \in \mathrm{Fix}(g_p) \}$. Since $\dim \mathrm{Fix}(g_p) \leq n-1$, it follows that

$$\mu_S(E_p^g) \leq (p^{n-1} - 1)/(p^n - 1) < 1/p.$$

Applying the Borel-Cantelli Lemma, since

$$\sum_{p \in T} \mu_S(E_p^g) < \sum_{p \in T} 1/p < \infty,$$

it follows that for μ_S-a.e. $v \in V_S$, the set $\{ p \in T \mid g_p \cdot v_p = v_p \}$ is finite. Hence $\mu_S(W_S^g) = 1$. □

Proof of Proposition 7.2. Suppose S, Z is a counterexample. Applying Lemma 7.3, we can suppose that $Z \subseteq V_S \cap \mathrm{Fr}_G(V)$. Let $G_S = [\bigoplus_{p \in S} T_p] \rtimes \Gamma$ and let ν_S be the uniform product probability measure on $V_S^+ = \prod_{p \in S} V(n,p)$. Then ν_S is a G_S-ergodic probability measure on V_S^+; and, identifying V_S with $\prod_{p \in S}(V(n,p) \smallsetminus \{ 0 \})$, we have that $\nu_S(V_S) = \prod_{p \in S}(1 - 1/p^n)$. Since $\sum_{p \in S} 1/p^n < \infty$, it follows that $\nu_S(V_S) > 0$ and hence $\nu_S(Z) = \mu_S(Z)\nu_S(V_S) > 0$. It follows that there exists $1 \neq t \in \bigoplus_{p \in S} T_p$ such that $tZ \cap Z \neq \emptyset$; say, $t \cdot v \in Z$, where $v \in Z$. Then, since v and $t \cdot v$ are E^*-equivalent, it follows that there exists $1 \neq g \in \Gamma$ such that

$g \cdot v = t \cdot v$. But then v and $g \cdot v$ are E_0-equivalent, which contradicts the fact that $v \in \mathrm{Fr}_G(V)$. □

This completes the proof of Theorem 1.3.

8. The proof of Lemma 7.1

In this section, we will present the proof of Lemma 7.1. Suppose that $f : V \to V$ is a counterexample; i.e. that $f : V \to V$ is a ν-nontrivial Borel homomorphism from E to E^* such that $\nu(\{ v \in V \mid f(v) \in \mathrm{Fr}_G(V) \}) \neq 1$. Then, by ergodicity, it follows that there exists a Γ-invariant Borel subset $X \subseteq V$ with $\nu(X) = 1$ such that $f(v) \notin \mathrm{Fr}_G(V)$ for all $v \in X$.

Let $E_0^{2^{\mathbb{P}}}$ be the countable Borel equivalence relation on $2^{\mathbb{P}}$ defined by

$$(x_p) \, E_0^{2^{\mathbb{P}}} \, (y_p) \quad \Longleftrightarrow \quad x_p = y_p \text{ for all but finitely many } p \in \mathbb{P}.$$

Then clearly $E_0^{2^{\mathbb{P}}} \cong E_0$. For each $v \in V$, let $\sigma(v) = \{ p \in \mathbb{P} \mid v_p \neq 0 \}$. Then the map $v \mapsto \sigma(v)$ is a Borel homomorphism from E to $E_0^{2^{\mathbb{P}}}$. Since Γ is a Kazhdan group, it follows that E is E_0-ergodic with respect to ν; and hence we can suppose that f maps X into a single $E_0^{2^{\mathbb{P}}}$-class; say, $[\, S\,]_{E_0^{2^{\mathbb{P}}}}$. Clearly S cannot be a finite subset of $\mathbb{P}$. Let $\mathcal{D}$ be *any* nonprincipal ultrafilter over $\mathbb{P}$ such that $S \in \mathcal{D}$.

From now on, for each $v \in V$, the corresponding E_0^V-class will be denoted by $[\, v\,]$; and we define

$$\Gamma_{[\, v\,]} = \{ \gamma \in \Gamma \mid \gamma \cdot [\, v\,] = [\, v\,] \}.$$

Let $v \in X$ and let $f(v) = (u_p)$. Then there exists a non-identity element $\gamma \in \Gamma \smallsetminus 1$ such that $\gamma \cdot [\, f(v)\,] = [\, \gamma \cdot f(v)\,] = [\, f(v)\,]$. Let

$$f(v)_{\mathcal{D}} = (u_p)_{\mathcal{D}} \in \prod\nolimits_{\mathcal{D}} V(n,p) = V(n,F),$$

where $F = \prod_{\mathcal{D}} \mathbb{F}_p$. For each $p \in \mathbb{P}$ and $\gamma \in \Gamma$, let $\gamma_p \in SL(n,p)$ be the corresponding matrix over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Then $(\gamma_p)_{\mathcal{D}} \cdot f(v)_{\mathcal{D}} = f(v)_{\mathcal{D}}$ and clearly $(\gamma_p)_{\mathcal{D}} = \gamma$. From now on, regard F as a subfield of $\mathbb{C}$ and let $\Gamma \curvearrowright V(n,\mathbb{C})$ be the corresponding extension of the action $\Gamma \curvearrowright V(n,F)$. Then $f(v)_{\mathcal{D}} \neq 0$ is a nontrivial element of the γ-eigenspace $E \leqslant V(n,\mathbb{C})$ corresponding to the eigenvalue 1; and since $\gamma \neq 1$, it follows that E is a nontrivial proper $\overline{\mathbb{Q}}$-subspace of $V(n,\mathbb{C})$.

Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$; and a subspace $W \leqslant V(n,\mathbb{C})$ is said to be a $\overline{\mathbb{Q}}$-subspace if there exists a (possibly empty) collection of vectors $\mathbf{b}_1, \dots, \mathbf{b}_t \in$

$V(n,\overline{\mathbb{Q}})$ such that $E = \langle \mathbf{b}_1, \dots, \mathbf{b}_t \rangle$. Clearly if E, $F \leqslant V(n,\mathbb{C})$ are $\overline{\mathbb{Q}}$-subspaces, then $E \cap F$ is also a $\overline{\mathbb{Q}}$-subspace. Thus, for each $v \in X$, there exists a unique minimal proper $\overline{\mathbb{Q}}$-subspace W_v such that $f(v)_{\mathcal{D}} \in W_v$. Furthermore, the above argument shows that each $\gamma \in \Gamma_{[\,f(v)\,]}$ acts trivially on W_v. Thus, for each $v \in X$, there exists a unique nontrivial maximal (necessarily proper) $\overline{\mathbb{Q}}$-subspace $U_v \leqslant V(n,\mathbb{C})$ on which $\Gamma_{[\,f(v)\,]}$ acts trivially. Note that the $\overline{\mathbb{Q}}$-subspace U_v does not depend on the choice of the nonprincipal ultrafilter over $\mathbb{P}$ such that $S \in \mathcal{D}$. In particular, the above argument shows that if $v \in X$ and if $\mathcal{D}$ is any nonprincipal ultrafilter over $\mathbb{P}$ such that $S \in \mathcal{D}$, then $f(v)_{\mathcal{D}} \in U_v$. Clearly the map $v \mapsto U_v$ is Borel and hence there exists a $\overline{\mathbb{Q}}$-subspace U such that

$$\nu(\{\, v \in X \mid U_v = U \,\}) > 0.$$

After adjusting f and shrinking X if necessary, we can suppose that $U_v = U$ for all $v \in X$. Let $H = \Gamma_{\{U\}}$ be the setwise stabilizer of U in Γ. Now suppose that v, $w \in X$ and that $\gamma \in \Gamma$ satisfies $\gamma \cdot v = w$. Then there exists $h \in \Gamma$ such that $h \cdot [\,f(v)\,] = [\,f(w)\,]$, and we have that

$$h[U] = h[U_v] = U_w = U.$$

Thus $h \in H$. Let $H_0 = \{\, h \restriction U \mid h \in H \,\}$ be the group of linear transformation induced on U by H; and for each $h \in H$, let $\tilde{h} = h \restriction U$.

Claim 8.1. *Suppose that v, $w \in X$ and that $h \in H$ satisfies $h \cdot [\,f(v)\,] = [\,f(w)\,]$. If $\tilde{h} = 1$, then $[\,f(v)\,] = [\,f(w)\,]$.*

Proof of Claim 8.1. Let $f(v) = (u_p)$ and $f(w) = (z_p)$. Suppose that

$$D = \{\, p \in \mathbb{P} \mid u_p \neq z_p \,\}$$

is infinite, and let $\mathcal{D}$ be a nonprincipal ultrafilter over $\mathbb{P}$ such that $D \in \mathcal{D}$. Then clearly $|D \smallsetminus S| < \infty$ and so $S \in \mathcal{D}$. Hence $f(v)_{\mathcal{D}}$, $f(w)_{\mathcal{D}} \in U$ and $f(v)_{\mathcal{D}} \neq f(w)_{\mathcal{D}}$. Since $h \cdot f(v)_{\mathcal{D}} = f(w)_{\mathcal{D}}$, it follows that $\tilde{h} \neq 1$. □

Again suppose that v, $w \in X$ and that $\gamma \in \Gamma$ satisfies $\gamma \cdot v = w$. Then there exists $h \in H$ such that $h \cdot [\,f(v)\,] = [\,f(w)\,]$. Suppose that $h' \in H$ also satisfies $h' \cdot [\,f(v)\,] = [\,f(w)\,]$. Then $h^{-1}h' \in \Gamma_{[\,f(v)\,]}$ and it follows that $h \restriction U = h' \restriction U$.

Hence we can define a Borel cocycle $\tilde{\alpha} : \Gamma \times X \to H_0$ by

$$\tilde{\alpha}(\gamma, v) = \tilde{h}, \text{ where } h \in H \text{ is any element such that } h \cdot [\, f(v) \,] = [\, f(\gamma \cdot v) \,].$$

By Theorem 4.7, there exists a subgroup $\Delta \leqslant \Gamma$ with $[\, \Gamma : \Delta \,] < \infty$ and an ergodic component $X_0 \subseteq X$ for the action $\Delta \curvearrowright X$ such that $\alpha \restriction (\, \Delta \times X_0 \,)$ is equivalent to either an embedding $\varphi : \Delta \to H_0$ or else the trivial homomorphism $\varphi : \Delta \to H_0$ which takes constant value 1. Furthermore, applying Theorem 4.9, it follows that Δ does not embed into H_0. Thus, after deleting a null subset of X_0 if necessary, we can suppose that there exists a Borel map $\tilde{b} : X_0 \to H_0$ such that

$$\tilde{b}(\gamma \cdot v)\tilde{\alpha}(\gamma, v)\tilde{b}(v)^{-1} = 1$$

for all $v \in X_0$ and $\gamma \in \Delta$. Let $b : X_0 \to H$ be a Borel map such that $\widetilde{b(v)} = \tilde{b}(v)$ and let $f' : X_0 \to V$ be the Borel map defined by $f'(v) = b(v) \cdot f(v)$.

Suppose that v, $w \in X_0$ and that $\gamma \in \Delta$ satisfies $\gamma \cdot v = w$. Let $h \in H$ be such that $h \cdot [\, f(v) \,] = [\, f(w) \,] = [\, f(\gamma \cdot v) \,]$. Then

$$b(\gamma \cdot v)hb(v)^{-1} \cdot [\, f'(v) \,] = b(\gamma \cdot v) \cdot [\, f(\gamma \cdot v) \,] = [\, f'(\gamma \cdot v) \,] = [\, f'(w) \,]$$

and $b(\gamma \cdot v)hb(v)^{-1} \mapsto \tilde{b}(\gamma \cdot v)\tilde{\alpha}(\gamma, v)\tilde{b}(v)^{-1} = 1$.

Claim 8.2. $[\, f'(v) \,] = [\, f'(w) \,]$.

Proof of Claim 8.2. Let $f'(v) = (u_p)$ and $f'(w) = (z_p)$. Suppose that

$$D = \{\, p \in \mathbb{P} \mid u_p \neq z_p \,\}$$

is infinite, and let $\mathcal{D}$ be a nonprincipal ultrafilter over $\mathbb{P}$ such that $D \in \mathcal{D}$. Once again, it is clear that $|D \smallsetminus S| < \infty$ and so $S \in \mathcal{D}$. Thus $f(v)_{\mathcal{D}}$, $f(w)_{\mathcal{D}} \in U$; and clearly $f'(v)_{\mathcal{D}} \neq f'(w)_{\mathcal{D}}$. Since $f'(v) = b(v) \cdot f(v)$ and $f'(w) = b(w) \cdot f(w)$ for some $b(v)$, $b(w) \in H = \Gamma_{\{U\}}$, it follows that $f'(v)_{\mathcal{D}}$, $f'(w)_{\mathcal{D}} \in U$. Finally, since

$$b(g \cdot v)hb(v)^{-1} \cdot [\, f'(v) \,] = [\, f'(w) \,],$$

it follows that $b(g \cdot v)hb(v)^{-1} \mapsto \tilde{b}(g \cdot v)\tilde{\alpha}(g, v)\tilde{b}(v)^{-1} \neq 1$, which is a contradiction. □

Since Δ is a Kazhdan group, it follows that $E_{\Delta}^{X_0}$ is E_0-ergodic; and hence we can suppose that f' maps X_0 into a single E_0^V-class. However, this implies that f

maps ν-a.e. $v \in X$ into a single E^*-class, which contradicts the assumption that f is ν-nontrivial. This completes the proof of Lemma 7.1.

References

[1] M. B. Bekka and M. Mayer, *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces*, London Math. Soc. Lecture Notes Series **269**, Cambridge University Press, 2000.

[2] L. Bowen, *Simple and large equivalence relations*, Proc. Amer. Math. Soc. **145** (2017), 215–224.

[3] R. Dougherty, S. Jackson and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193–225.

[4] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[5] J. Feldman, C. E. Sutherland, and R. J. Zimmer, *Subrelations of ergodic equivalence relations*, Ergodic Theory Dynam. Systems **9** (1989), 239–269.

[6] P. de la Harpe and A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)*, Astérisque **175** (1989).

[7] A. Ioana, *Cocycle Superrigidity for Profinite Actions of Property (T) Groups*, Duke Math. J. **157** (2011), 337–367.

[8] S. Jackson, A.S. Kechris, and A. Louveau, Countable Borel equivalence relations, J. Math. Logic 2 (2002) 1–80.

[9] A. Karras and D. Solitar, *The free product of two groups with a malnormal amalgamated subgroup*, Can. J. Math. **23** (1971), 933–959.

[10] A. S. Kechris and B. D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics **1852**, Springer-Verlag, Berlin, 2004.

[11] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Erg. der Math. und ihrer Grenz., **17**, Springer-Verlag, 1991.

[12] M. G. Nadkarni, *On the existence of a finite invariant measure*, Proc. Indian Acad. Sci. Math. Sci. **100** (1991), 203–220.

[13] M. G. Nadkarni, *Orbit equivalence and Kakutani equivalence in descriptive setting*, preprint, 1991.

[14] A. Yu. Ol'shanskii, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Internat. J. Algebra Comput. **3** (1993), 365–409.

[15] N. Ozawa, *There is no separable universal* II_1*-factor* Proc. Amer. Math. Soc. **132** (2004), 487–490.

[16] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, Invent. Math. **170** (2007), 243–295.

[17] S. Thomas, *Superrigidity and countable Borel equivalence relations*, Ann. Pure Appl. Logic **120** (2003), 237–262.

[18] S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*, J. Amer. Math. Soc. **16** (2003), 233–258.

[19] S. Thomas, *Popa superrigidity and countable Borel equivalence relations*, Annals Pure Appl. Logic. **158** (2009), 175–189.

[20] S. Thomas, *The classification problem for S-local torsion-free abelian groups of finite rank*, Adv. Math. **226** (2011), 3699–3723.

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854

E-mail address: simon.rhys.thomas@gmail.com