CHARACTERS OF INDUCTIVE LIMITS OF FINITE ALTERNATING GROUPS

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ABSTRACT. If $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an inductive limit of finite alternating groups, then the indecomposable characters of G are precisely the associated characters of the ergodic invariant random subgroups of G.

1. Introduction

In [17], Vershik pointed out that the indecomposable characters of the group $\mathrm{Fin}(\mathbb{N})$ of finitary permutations of the natural numbers are closely connected with its ergodic invariant random subgroups; and in [16], he suggested that this should also be true of various other locally finite groups. In this paper, we will prove that if $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an inductive limit of finite alternating groups, then the indecomposable characters of G are precisely the associated characters of the ergodic invariant random subgroups of G.

Let G be a countably infinite group and let Sub_G be the compact space of subgroups $H \leqslant G$. Then a Borel probability measure ν on Sub_G which is invariant under the conjugation action of G on Sub_G is called an *invariant random subgroup* or *IRS*. For example, suppose that G acts via measure-preserving maps on the Borel probability space (Z, μ) and let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant map defined by

$$z \mapsto G_z = \{ g \in G \mid g \cdot z = z \}.$$

Then the corresponding *stabilizer distribution* $\nu = f_*\mu$ is an IRS of G. In fact, by a result of Abért-Glasner-Virag [1], every IRS of G can be realized as the stabilizer distribution of a suitably chosen measure-preserving action. Moreover, by Creutz-Peterson [4], if ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

If G is a countable group, then a function $\chi : G \to \mathbb{C}$ is said to be a *character* if the following conditions are satisfied:

(i) $\chi(h\,g\,h^{-1}) = \chi(g)$ for all $g, \in G$.
(ii) $\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ and $g_1, \cdots, g_n \in G$.
(iii) $\chi(1_G) = 1$.

For example, if $G \curvearrowright (Z, \mu)$ is any measure-preserving action on a Borel probability space, then we can define a character χ of G by $\chi(g) = \mu(\mathrm{Fix}_Z(g))$. In particular, if ν is an IRS of G, then we can define a corresponding character χ by

$$\begin{aligned}\chi(g) &= \nu(\{ H \in \mathrm{Sub}_G \mid gHg^{-1} = H \}) \\ &= \nu(\{ H \in \mathrm{Sub}_G \mid g \in N_G(H) \}).\end{aligned}$$

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On the other hand, we can also define a second character χ' by

$$\chi'(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \}).$$

It is easily seen that $\chi' = \chi$ if and only if $N_G(H) = H$ for ν-a.e. $H \in \mathrm{Sub}_G$. Fortunately, if $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an inductive limit of finite alternating groups, then this is true of every ergodic IRS ν of G, except for the Dirac measure δ_1 which concentrates on the identity subgroup 1. (This result is proved during the proof of Thomas-Tucker-Drob [15, Theorem 3.21].) Since it turns out to be slightly more convenient to work with the character χ', we choose the following definition.

Definition 1.1. If ν is an IRS of the countable group G, then the *associated character* χ_ν is defined to be $\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \})$.

A character χ is said to be *indecomposable* or *extremal* if it is impossible to express $\chi = r\chi_1 + (1-r)\chi_2$, where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters. The set of characters of G will be denoted by $\mathcal{F}(G)$ and the set of indecomposable characters will be denoted by $\mathcal{E}(G)$. The set $\mathcal{F}(G)$ always contains the two *trivial* characters χ_{con} and χ_{reg}, where $\chi_{\mathrm{con}}(g) = 1$ for all $g \in G$ and $\chi_{\mathrm{reg}}(g) = 0$ for all $1 \neq g \in G$. It is well-known that χ_{con} is indecomposable, and that χ_{reg} is indecomposable if and only if G is an i.c.c. group, i.e. the conjugacy class g^G of every nonidentity element $g \in G$ is infinite. (For example, see Peterson-Thom [10].) Let δ_G and δ_1 be the Dirac measures which concentrate on the normal subgroups G, 1 respectively. Then δ_1, δ_G are ergodic IRSs of G and clearly $\chi_{\mathrm{con}} = \chi_{\delta_G}$ and $\chi_{\mathrm{reg}} = \chi_{\delta_1}$. Throughout this paper, we will refer to δ_G, δ_1 as the *trivial* ergodic IRSs of G.

Definition 1.2. A simple locally finite group G is said to be an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of a strictly increasing chain of finite alternating groups G_i. (Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.)

We are now in a position to state the main result of this paper.

Theorem 1.3. *If G is an $L(\mathrm{Alt})$-group and $G \not\cong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are* precisely *the associated characters χ_ν of the ergodic invariant random subgroups ν of G.*

Note that the statement of Theorem 1.3 makes two distinct assertions about the characters of the $L(\mathrm{Alt})$-group $G \not\cong \mathrm{Alt}(\mathbb{N})$. Firstly, if ν is any ergodic IRS of G, then the associated character χ_ν is indecomposable; and, secondly, that every indecomposable character of G is the associated character χ_ν of some ergodic IRS ν of G. The former statement was proved in Thomas-Tucker-Drob [15], and so it will be enough for us to prove the latter statement in this paper. Also note that [15] contains a classification of the ergodic IRSs of the $L(\mathrm{Alt})$-group $G \not\cong \mathrm{Alt}(\mathbb{N})$. Thus, combining the results of this paper and [15], we obtain a classification of the indecomposable characters of the $L(\mathrm{Alt})$-group $G \not\cong \mathrm{Alt}(\mathbb{N})$. Of course, the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$ have already been classified by Thoma [14]. (It is perhaps interesting to note that both of the assertions in Theorem 1.3 fail when $G = \mathrm{Alt}(\mathbb{N})$.)

The indecomposable characters of the diagonal limits $G = \bigcup_{i \in \mathbb{N}} G_i$ of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ such that $G \not\cong \mathrm{Alt}(\mathbb{N})$ were earlier classified by Leinen-Puglisi [7]. (Recall that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a *diagonal limit* if for each $i \in \mathbb{N}$, every orbit of G_i on Δ_{i+1} is either natural or trivial.) It should be stressed that the proof of Theorem 1.3 makes essential use of the ideas and techniques of Leinen-Puglisi [7].

This paper is organized as follows. In Section 2, we will briefly discuss the ergodic IRSs of the $L(\mathrm{Alt})$-groups; and in Section 3, we will briefly discuss the irreducible characters of the finite alternating groups. In Sections 4 and 5, we will present the proof of Theorem 1.3. In Section 6, we will point out how both of the assertions in Theorem 1.3 fail when $G = \mathrm{Alt}(\mathbb{N})$.

Finally, we will explain our notation for the various kinds of limits that arise in this paper. Suppose that $(r_i \mid i \in \mathbb{N})$ is a bounded sequence of real numbers. If $I \subseteq \mathbb{N}$ is an infinite subset which is enumerated in increasing order by the sequence $(i_k \mid k \in \mathbb{N})$, then we will write $\lim_{i \in I} r_i$ instead of $\lim_{k\to\infty} r_{i_k}$. Also if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then $\lim_{\mathcal{U}} r_i$ will denote the unique real number r such that $\{ i \in \mathbb{N} : |r_i - r| < \varepsilon \} \in \mathcal{U}$ for all $\varepsilon > 0$.

2. The ergodic IRSs of the $L(\mathrm{Alt})$-groups

In this section, we will present a brief discussion of the ergodic IRSs of the $L(\mathrm{Alt})$-groups. First we need to introduce some notation. Suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. For each $i \in \mathbb{N}$, let

- $n_i = |\Delta_i|$;
- s_{i+1} be the number of natural G_i-orbits on Δ_{i+1};
- f_{i+1} be the number of trivial G_i-orbits on Δ_{i+1};
- $e_{i+1} = n_{i+1} - (s_{i+1}n_i + f_{i+1})$ is the number of points $x \in \Delta_{i+1}$ which lie in a nontrivial non-natural G_i-orbit.

Here an orbit Ω of $G_i = \mathrm{Alt}(\Delta_i)$ on Δ_{i+1} is said to be *natural* if $|\Omega| = |\Delta_i|$ and the action $G_i \curvearrowright \Omega$ is isomorphic to the natural action $G_i \curvearrowright \Delta_i$. Also for each $i < j$, let $s_{ij} = s_{i+1}s_{i+2}\cdots s_j$. Thus s_{ij} is the number of "obvious" natural orbits of G_i on Δ_j.

The classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups involves a fundamental dichotomy which was introduced by Leinen-Puglisi [6, 7] in the more restrictive setting of diagonal limits of finite alternating groups, i.e. the linear vs sublinear natural orbit growth condition.

Lemma 2.1 (Leinen-Puglisi [7]). *For each $i \in \mathbb{N}$, the limit $a_i = \lim_{j\to\infty} s_{ij}/n_j$ exists.*

Definition 2.2. An $L(\mathrm{Alt})$-groups $G = \bigcup_{i\in\mathbb{N}} G_i$ has *linear natural orbit growth* if $a_i > 0$ for some $i \in \mathbb{N}$. Otherwise, $G = \bigcup_{i\in\mathbb{N}} G_i$ has *sublinear natural orbit growth.*

Remark 2.3. Clearly if $G = \bigcup_{i\in\mathbb{N}} G_i$ has linear natural orbit growth, then there exists $i_0 \in \mathbb{N}$ such that $s_{i+1} > 0$ for all $i \geq i_0$. Also since $a_i = s_{i+1}a_{i+1}$, it follows that $a_i > 0$ for every $i \geq i_0$.

Since the proof of Theorem 1.3 makes use of the classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups of linear natural orbit growth, we will briefly describe this classification. So suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ has linear natural orbit growth. Then, after replacing the increasing union $G = \bigcup_{i\in\mathbb{N}} G_i$ by $G = \bigcup_{i_0\leq i\in\mathbb{N}} G_i$ for some suitably chosen $i_0 \in \mathbb{N}$, we can suppose that $s_{i+1} > 0$ for all $i \in \mathbb{N}$. Let $t_0 = n_0$ and let $t_{i+1} = n_{i+1} - s_{i+1}n_i$. Then we can suppose that:

- $\Delta_0 = \{ \alpha^0_\ell \mid \ell < t_0 \}$; and

- $\Delta_{i+1} = \{ \sigma^\frown k \mid \sigma \in \Delta_i, 0 \leq k < s_{i+1} \} \cup \{ \alpha_\ell^{i+1} \mid 0 \leq \ell < t_{i+1} \}$;

and that the embedding $\varphi_i : \mathrm{Alt}(\Delta_i) \to \mathrm{Alt}(\Delta_{i+1})$ satisfies

$$\varphi_i(g)(\sigma^\frown k) = g(\sigma)^\frown k$$

for each $\sigma \in \Delta_i$ and $0 \leq k < s_{i+1}$. Let Δ consist of all sequences of the form $(\alpha_\ell^i, k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$, where $i \in \mathbb{N}$ and k_j is an integer such that $0 \leq k_j < s_j$. For each $i \in \mathbb{N}$ and $\sigma \in \Delta_i$, let $\Delta(\sigma) \subseteq \Delta$ be the subset of sequences of the form $\sigma \,^\frown (k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$. Then the sets $\Delta(\sigma)$ form a clopen basis for a locally compact topology on Δ; and by Thomas-Tucker-Drob [15, Proposition 3.18], there exists a unique G-invariant ergodic probability measure m on Δ. By Thomas-Tucker-Drob [15, Corollary 2.5], since G is a simple locally finite group, it follows that the product action $G \curvearrowright (\Delta^r, m^{\otimes r})$ is also ergodic for all $r \geq 2$, and hence the corresponding stabilizer distribution ν_r is an ergodic IRS of G.

Theorem 2.4 (Thomas-Tucker-Drob [15])**.** *If $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth, then the ergodic IRSs of G are $\{ \delta_1, \delta_G \} \cup \{ \nu_r \mid r \in \mathbb{N}^+ \}$.*

From now on, whenever $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth, then we will refer to $G \curvearrowright (\Delta, m)$ as the *canonical ergodic action*. Since the proof of Theorem 1.3 does not require any knowledge of the ergodic IRSs of $L(\mathrm{Alt})$-group of sublinear natural orbit growth, we refer the interested reader to Thomas-Tucker-Drob [15] for the statements of the classification theorems. (The cases when $G \ncong \mathrm{Alt}(\mathbb{N})$ and $G \cong \mathrm{Alt}(\mathbb{N})$ need to be handled separately.)

3. Irreducible characters of finite alternating groups

In this section, we will discuss some results of Leinen-Puglisi [7] concerning the asymptotic properties of the irreducible characters of $\mathrm{Alt}(n)$. But first, following Zalesskii [19], we will discuss the relationship between the irreducible characters of $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$. It is well-known that the irreducible representations of the symmetric group $\mathrm{Sym}(n)$ are parametrized by the partitions $\lambda = (\ell_1, \ell_2, \cdots \ell_r)$ of n; i.e. sequences of integers such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r > 0$ and $\ell_1 + \ell_2 + \cdots + \ell_r = n$. For each such partition λ, let φ_λ be the corresponding irreducible character of $\mathrm{Sym}(n)$ and let D_λ be the corresponding Young diagram. Thus D_λ is an array of cells with ℓ_k cells in the kth row for each $1 \leq k \leq r$. Also let λ^* be the partition corresponding to the Young diagram obtained from D_λ by reflection in the diagonal that runs rightwards and downwards from the upper left-hand corner of D_λ. For example, $(5, 2, 1)^* = (3, 2, 1, 1, 1)$. Finally, let $\trianglelefteq$ and $\leq$ be the dominance and lexicographic orders on the set of partitions of n. (For example, see Sagan [12].)

If λ is a partition of n such that $\lambda \neq \lambda^*$, then $\varphi_\lambda \restriction \mathrm{Alt}(n)$ is an irreducible character of $\mathrm{Alt}(n)$, which is equal to $\varphi_{\lambda^*} \restriction \mathrm{Alt}(n)$. On the other hand, if $\lambda = \lambda^*$, then $\varphi_\lambda \restriction \mathrm{Alt}(n)$ is the sum of two distinct irreducible representations of $\mathrm{Alt}(n)$. Furthermore, for every irreducible character θ of $\mathrm{Alt}(n)$, there exists a unique λ such that $\lambda \geq \lambda^*$ and θ is an irreducible component of $\varphi_\lambda \restriction \mathrm{Alt}(n)$. This allows us to associate a partition λ such that $\lambda \geq \lambda^*$ with each irreducible character θ of $\mathrm{Alt}(n)$. If $\lambda > \lambda^*$, then λ is associated with a unique irreducible character of $\mathrm{Alt}(n)$; while if $\lambda = \lambda^*$, then λ is associated with a pair of irreducible characters of $\mathrm{Alt}(n)$. If λ is associated with the irreducible character θ of $\mathrm{Alt}(n)$, then we write $D(\theta) = D_\lambda$ for the corresponding Young diagram. For later use, note that since $\lambda \geq \lambda^*$, it follows that the length of the first row of each Young diagram $D(\theta)$ is greater or equal to the length of the first column.

For each partition $\lambda = (\, \ell_1, \ell_2, \cdots \ell_r \,)$ of n such that $\lambda \geq \lambda^*$, we define its *type* to be $\alpha_\lambda = (\, \ell_2, \cdots \ell_r \,)$ and its *depth* to be $d(\lambda) = \ell_2 + \cdots + \ell_r$. Similarly, we will refer to the types and depths of the corresponding Young diagrams and the corresponding irreducible characters of $\mathrm{Alt}(n)$; and if $\alpha = (\, \ell_2, \cdots \ell_r \,)$ is a type, then we will refer to $d(\alpha) = \ell_2 + \cdots + \ell_r$ as its depth. Of course, since $\ell_1 = n - d(\alpha)$, the corresponding partition λ_α of n is uniquely determined by α; and if $n \geq 2d(\alpha) + 1$, then $\lambda_\alpha > \lambda_\alpha^*$ and so there exists a unique irreducible character of $\mathrm{Alt}(n)$ of type α, which we will denote by θ_α. Finally, for each integer $n \geq 2d(\alpha)+1$, let Φ_α be the set of partitions $(\, P_1, P_2, \cdots, P_r \,)$ of n such that $|P_1| = n - d(\alpha)$ and $|P_k| = \ell_k$ for each $2 \leq k \leq r$, and let π_α be the permutation character of the action $\mathrm{Alt}(n) \curvearrowright \Phi_\alpha$. In the remainder of this section, we will present some results of of Leinen-Puglisi [7] concerning the asymptotic properties of θ_α and π_α for some fixed type α as $n \to \infty$. We will be begin by stating two results concerning the growth rates of the degrees $\pi_\alpha(1)$, $\theta_\alpha(1)$ of the representations. The first result is an easy exercise. For a proof of the second result, see Leinen-Puglisi [7, Lemma 3.1].

Lemma 3.1. *For each type α, there exists a polynomial $p_\alpha \in \mathbb{Q}[x]$ of degree $d(\alpha)$ such that if $n \geq 2d(\alpha) + 1$, then $p_\alpha(n) = \pi_\alpha(1) = |\Phi_\alpha|$ is the degree of the permutation character π_α of the action $\mathrm{Alt}(n) \curvearrowright \Phi_\alpha$.*

Lemma 3.2. *For each type α, there exists a polynomial $q_\alpha \in \mathbb{Q}[x]$ of degree $d(\alpha)$ such that if $n \geq 2d(\alpha)+1$, then $q_\alpha(n) = \theta_\alpha(1)$ is the degree of the unique irreducible character θ_α of $\mathrm{Alt}(n)$ of type α.*

Before we can state the final result of this section, we first need to translate the dominance order on partitions to a corresponding partial order on types. So suppose that α, β are types. Let n be an integer such that $n \geq \max\{\, 2d(\alpha) + 1, 2d(\beta) + 1 \,\}$ and let λ_α, λ_β be the corresponding partitions of n. Then we define

$$\alpha \trianglelefteq \beta \quad \Longleftrightarrow \quad \lambda_\alpha \trianglelefteq \lambda_\beta.$$

It is easily checked that this definition is independent of the choice of the integer $n \geq \max\{\, 2d(\alpha) + 1, 2d(\beta) + 1 \,\}$. The following result, which is extracted from the proof of Leinen-Puglisi [7, Theorem 3.2], will play a key role in the next section. For the sake of completeness, we will sketch the main points of its proof.

Lemma 3.3. *Let α be a type of depth $d = d(\alpha)$, let n be an integer such that $n \geq 2d + 1$, and let θ_α be the irreducible character of $\mathrm{Alt}(n)$ of type α. Then there exist integers $z_\beta \in \mathbb{Z}$, which are are independent of n, such that*

$$\theta_\alpha = \sum_{\beta \trianglerighteq \alpha} z_\beta \pi_\beta. \tag{3.3.a}$$

Furthermore, the integers z_β satisfy:

$$\lim_{n \to \infty} \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta) = d}} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} = 1. \tag{3.3.b}$$

Sketch proof. Suppose that λ is any partition of n such that $n \geq 2d(\lambda) + 1$. If σ is any partition of n such that $\sigma \trianglerighteq \lambda$, then $d(\sigma) \leq d(\lambda)$ and so $n \geq 2d(\sigma) + 1$. In particular, letting φ_σ be the corresponding irreducible character of $\mathrm{Sym}(n)$, we have that $\varphi_\sigma \upharpoonright \mathrm{Alt}(n)$ is the unique irreducible character θ_σ associated with σ.

Thus Young's rule [12, Theorem 2.11.2] implies that

$$\theta_\lambda = \pi_\lambda - \sum_{\sigma \rhd \lambda} \kappa_{\sigma\lambda}\theta_\sigma, \tag{3.1}$$

where $\kappa_{\sigma\lambda}$ is the corresponding Kostka number; i.e. the number of semi-standard tableaux of shape σ and content λ. It is easily checked that, since

$$n \geq 2d(\sigma) + 1 \geq 2d(\lambda) + 1,$$

each of these Kostka numbers $\kappa_{\sigma\lambda}$ depends only on the types of σ and λ. In particular, letting λ be the partition of n corresponding to the type α, we can replace each partition in (3.1) by its corresponding type, and so obtain the following equality:

$$\theta_\alpha = \pi_\alpha - \sum_{\beta \rhd \alpha} \kappa_{\beta\alpha}\theta_\beta.$$

Proceeding inductively along the dominance order for types, we now easily obtain equation (3.3.a). In particular, we have that

$$\theta_\alpha(1) = \sum_{\beta \trianglerighteq \alpha} z_\beta \pi_\beta(1)$$

and so

$$1 = \sum_{\beta \trianglerighteq \alpha} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)}.$$

Using Lemmas 3.1 and 3.2, we easily obtain equation (3.3.b). □

4. Full limits of finite alternating groups

In this section, we will prove Theorem 1.3 in the special case when $G = \bigcup_{i\in\mathbb{N}} G_i$ is a "full limit" of finite alternating groups. Our arguments in the first half of this section will follow those of Leinen-Puglisi [7, Section 3].

Definition 4.1. Suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$.

(i) The embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is said to be *full* if $\mathrm{Alt}(\Delta_i)$ has no trivial orbits on Δ_{i+1}.

(ii) $G = \bigcup_{i\in\mathbb{N}} G_i$ is the *full limit* of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ if every embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is full.

Warning 4.2. A composition of two full embeddings is not necessarily full. Consequently, if $G = \bigcup_{i\in\mathbb{N}} G_i$ is a full limit and $(k_i \mid i \in \mathbb{N})$ is a strictly increasing sequence of natural numbers, then $G = \bigcup_{i\in\mathbb{N}} G_{k_i}$ is not necessarily a full limit. The notion of a full limit is a purely technical one, first introduced in Thomas-Tucker-Drob [15], which is useful in the proofs of results about $L(\mathrm{Alt})$-groups.

Most of this section will be devoted to the proof of the following result.

Proposition 4.3. *Suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is the full limit of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and that G has a nontrivial indecomposable character χ. Then:*

(a) *$\chi = \chi_\nu$ is the associated character of a nontrivial ergodic IRS ν of G; and*
(a) *$G = \bigcup_{i\in\mathbb{N}} G_i$ has linear natural orbit growth.*

The proof of Proposition 4.3 will make use of the following result.

Proposition 4.4 (Thomas-Tucker-Drob [15])**.** *If $G = \bigcup_{i\in\mathbb{N}} G_i$ is the full limit of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, then G has a nontrivial ergodic IRS if and only if $G = \bigcup_{i\in\mathbb{N}} G_i$ has linear natural orbit growth.*

We will also make use of the following result, which is slight reformulation of Thomas-Tucker-Drob [15, Corollary 7.5].

Lemma 4.5. *If $G = \bigcup_{i\in\mathbb{N}} G_i$ is the full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, then $\liminf |\operatorname{supp}_{\Delta_i}(g)|/|\Delta_i| > 0$ for all $1 \neq g \in G$.*

From now on, suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is the full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and that χ is a nontrivial indecomposable character of G. Then, by Vershik-Kerov [18, Theorem 6], there exist irreducible characters θ_i of G_i such that for all $g \in G$,

$$\chi(g) = \lim_{i\to\infty} \widehat{\theta}_i(g),$$

where $\widehat{\theta}_i = \theta_i/\theta_i(1)$ is the corresponding normalized irreducible character. For each $i \in \mathbb{N}$, let d_i be the depth of the corresponding Young diagram $D(\theta_i)$. The proof of the next lemma is almost identical to that of Leinen-Puglisi [7, Proposition 3.5].

Lemma 4.6. $\limsup d_i < \infty$.

Proof. Since $\chi \neq \chi_{\mathrm{reg}}$, there exists a nonidentity element $1 \neq g \in G$ such that $\chi(g) \neq 0$. Applying Lemma 4.5, there exists $c > 0$ such that $|\operatorname{supp}_{\Delta_i}(g)| \geq cn_i$ for all sufficiently large i. Also, by Roichman [11, Theorem 5.4], since the length $n_i - d_i$ of the first row of the Young diagram $D(\theta_i)$ is greater or equal to the length of the first column, it follows that there exist constants $b > 0$ and $0 < q < 1$ such that if i is sufficiently large, then

$$|\widehat{\theta}_i(g)| \leq \left(\max\left\{\, q, \frac{n_i - d_i}{n_i} \,\right\}\right)^{b\cdot|\operatorname{supp}_{\Delta_i}(g)|}.$$

Since $\chi(g) = \lim_{i\to\infty} \widehat{\theta}_i(g) \neq 0$ and $\lim_{i\to\infty} |\operatorname{supp}_{\Delta_i}(g)| = \infty$, it follows that if i is sufficiently large, then $\max\{\, q, (n_i - d_i)/n_i \,\} = (n_i - d_i)/n_i$ and so

$$|\widehat{\theta}_i(g)| \leq \left(\frac{n_i - d_i}{n_i}\right)^{b\cdot|\operatorname{supp}_{\Delta_i}(g)|} = \left(1 - \frac{d_i}{n_i}\right)^{b\cdot|\operatorname{supp}_{\Delta_i}(g)|}$$

It also now follows that $d_i/n_i \to 0$ as $i \to \infty$. Since $|\operatorname{supp}_{\Delta_i}(g)| \geq cn_i$ for all sufficiently large i, we have that

$$|\widehat{\theta}_i(g)| \leq \left(\left(1 - \frac{d_i}{n_i}\right)^{\frac{n_i}{d_i}}\right)^{bcd_i}$$

Since $d_i/n_i \to 0$, it follows that

$$\left(1 - \frac{d_i}{n_i}\right)^{\frac{n_i}{d_i}} \to \left(\frac{1}{e}\right)$$

and this implies that $\limsup d_i < \infty$. $\square$

Thus there exists an infinite subset $I \subseteq \mathbb{N}$ such that the irreducible character θ_i has the same type α for each $i \in I$. Let $d = d(\alpha)$ be the corresponding depth.

Lemma 4.7. $\chi(g) = \lim_{i\in I}(|\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i|)^d$ *for all $g \in G$.*

Proof. Suppose that $i \in I$ and that $n_i \gg d$. In order to simplify notation, we will write n, Δ, G, θ instead of n_i, Δ_i, G_i, θ_i and we will write limits as $\lim_{n\to\infty}$ instead of $\lim_{i\in I}$. For each type $\beta = (\,\ell_2, \cdots \ell_r\,) \trianglerighteq \alpha$, let $d(\beta)$ be the corresponding depth and let Φ_β be the corresponding set of partitions $(\,P_1, P_2, \cdots, P_r\,)$ of Δ such that $|P_1| = n - d(\beta)$ and $|P_k| = \ell_k$ for each $2 \leq k \leq r$. Let π_β be the permutation character of the action $G \curvearrowright \Phi_\beta$ and let $\widehat{\pi}_\beta = \pi_\beta/\pi_\beta(1)$ be the corresponding normalized permutation character.

Claim 4.8. *For each type $\beta = (\,\ell_2, \cdots \ell_r\,) \trianglerighteq \alpha$ and element $g \in G$,*

$$\lim_{n\to\infty} \widehat{\pi}_\beta(g) = \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^{d(\beta)}$$

Proof of Claim 4.8. Clearly we can suppose that $g \neq 1$. Let

$$F_0(g) = \{\,(\,P_1, P_2, \cdots, P_r\,) \in \operatorname{Fix}_{\Phi_\beta}(g) \mid \operatorname{supp}_\Delta(g) \subseteq P_1\,\}$$

and let $F_1(g) = \operatorname{Fix}_{\Phi_\beta}(g) \smallsetminus F_0(g)$. Let c_β be the number of partitions of a $d(\beta)$-set into pieces of sizes $\ell_2, \cdots, \ell_r$. Then clearly

$$|\Phi_\beta| = c_\beta \binom{n}{d(\beta)} \qquad \text{and} \qquad |F_0(g)| = c_\beta \binom{|\operatorname{Fix}_\Delta(g)|}{d(\beta)}.$$

If $(\,P_1, P_2, \cdots, P_r\,) \in F_1(g)$, then $P_2 \sqcup \cdots \sqcup P_r$ is the union of s nontrivial g-orbits $\sigma_1, \cdots, \sigma_s$ and $t = d(\beta) - \sum_{j=1}^{s} |\sigma_j|$ trivial g-orbits for some $1 \leq s \leq d(\beta)/2$. Clearly $0 \leq t \leq d(\beta) - 2s$. Since g obviously has less than n nontrivial orbits, it follows that

$$\begin{aligned} |F_1(g)| &< c_\beta \sum_{s=1}^{d(\beta)/2} \binom{n}{s} \sum_{t=0}^{d(\beta)-2s} \binom{|\operatorname{Fix}_\Delta(g)|}{t} \\ &< c_\beta \sum_{s=1}^{d(\beta)/2} \binom{n}{s} \sum_{t=0}^{d(\beta)-2s} \binom{n}{t} \end{aligned}$$

and so there exists a polynomial $q(x) \in \mathbb{Z}[x]$ of degree at most $d(\beta) - 1$ such that $|F_1(g)| < q(n)$. Since $|\Phi_\beta|$ is a polynomial function of degree $d(\beta)$, it follows that $\lim_{n\to\infty} |F_1(g)|/|\Phi_\beta| = 0$. Hence

$$\begin{aligned} \lim_{n\to\infty} \widehat{\pi}_\beta(g) &= \lim_{n\to\infty} |F_0(g)|/|\Phi_\beta| \\ &= \lim_{n\to\infty} \binom{|\operatorname{Fix}_\Delta(g)|}{d(\beta)} \Big/ \binom{n}{d(\beta)} \\ &= \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^{d(\beta)}. \end{aligned}$$

□

Recall that $d(\alpha) = d$. Hence, applying Lemma 3.3, there exist integers $z_\beta \in \mathbb{Z}$, which are independent of n, such that

$$\theta = \theta_\alpha = \sum_{\beta \trianglerighteq \alpha} z_\beta \pi_\beta \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta) = d}} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} = 1.$$

It follows that for each $g \in G$,

$$\begin{aligned}
\chi(g) = \lim_{n\to\infty} \widehat{\theta}_\alpha(g) &= \lim_{n\to\infty} \sum_{\beta \trianglerighteq \alpha} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} \widehat{\pi}_\beta(g) \\
&= \sum_{\beta \trianglerighteq \alpha} z_\beta \lim_{n\to\infty} \frac{\pi_\beta(1)}{\theta_\alpha(1)} \lim_{n\to\infty} \widehat{\pi}_\beta(g) \\
&= \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta)=d}} z_\beta \lim_{n\to\infty} \frac{\pi_\beta(1)}{\theta_\alpha(1)} \lim_{n\to\infty} \widehat{\pi}_\beta(g) \\
&= \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta)=d}} z_\beta \lim_{n\to\infty} \frac{\pi_\beta(1)}{\theta_\alpha(1)} \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^d \\
&= \left(\lim_{n\to\infty} \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta)=d}} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} \right) \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^d \\
&= \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^d.
\end{aligned}$$

This completes the proof of Lemma 4.7. □

For each $i \in \mathbb{N}$, let $\Omega_i = \Delta_i^d$ and let $G_i \curvearrowright \Omega_i$ be the product action. Then the corresponding normalized permutation character of G_i is

$$|\operatorname{Fix}_{\Omega_i}(g)|/|\Omega_i| = (|\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i|)^d;$$

and hence for each $g \in G$, we have that $\chi(g) = \lim_{i\in I} |\operatorname{Fix}_{\Omega_i}(g)|/|\Omega_i|$. We are now ready to complete the proof of Proposition 4.3. Our argument makes use of the Loeb measure construction [9]. Our exposition and notation follow that of Conley-Kechris-Tucker-Drob [3].

For each $i \in \mathbb{N}$, let μ_i be the uniform probability measure on Ω_i defined by $\mu_i(A) = |A|/|\Omega_i|$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $I \in \mathcal{U}$ and let $\sim_\mathcal{U}$ be the equivalence relation on $X = \prod_{i\in\mathbb{N}} \Omega_i$ defined by

$$(x_i) \sim_\mathcal{U} (y_i) \iff \{\, i \in \mathbb{N} \mid x_i = y_i \,\} \in \mathcal{U}.$$

For each $(x_i) \in X$, let $[(x_i)]_\mathcal{U}$ be the corresponding $\sim_\mathcal{U}$-equivalence class, and let

$$X_\mathcal{U} = \{\, [(x_i)]_\mathcal{U} \mid (x_i) \in X \,\}.$$

For each sequence $(A_i) \in \prod_{i\in\mathbb{N}} \mathcal{P}(\Omega_i)$, define the subset $[(A_i)]_\mathcal{U} \subseteq X_\mathcal{U}$ by

$$[(x_i)]_\mathcal{U} \in [(A_i)]_\mathcal{U} \iff \{\, i \in \mathbb{N} \mid x_i \in A_i \,\} \in \mathcal{U}.$$

Then $\mathbf{B}^0_\mathcal{U} = \{\, [(A_i)]_\mathcal{U} \mid (A_i) \in \prod_{i\in\mathbb{N}} \mathcal{P}(\Omega_i) \,\}$ is a Boolean algebra of subsets of $X_\mathcal{U}$, and we can define a finitely additive probability measure $\mu_\mathcal{U}$ on $\mathbf{B}^0_\mathcal{U}$ by

$$\mu_\mathcal{U}([(A_i)]_\mathcal{U}) = \lim\nolimits_\mathcal{U} \mu_i(|A_i|).$$

Furthermore, there exists a σ-algebra $\mathbf{B}_\mathcal{U}$ of subsets of $X_\mathcal{U}$ such that $\mathbf{B}^0_\mathcal{U} \subseteq \mathbf{B}_\mathcal{U}$ and such that $\mu_\mathcal{U}$ extends to a σ-additive probability measure on $\mathbf{B}_\mathcal{U}$, which we will also denote by $\mu_\mathcal{U}$. (A clear account of the construction of $\mathbf{B}_\mathcal{U}$ and $\mu_\mathcal{U}$ can be found in Conley-Kechris-Tucker-Drob [3, Section 2].) Thus $(\, X_\mathcal{U}, \mathbf{B}_\mathcal{U}, \mu_\mathcal{U} \,)$ is a probability space. (Although this will not cause any difficulties in this proof, it is perhaps still worthwhile to note that this probability space is non-separable.)

Next for each $g \in G$ and $x \in \Omega_i$, we define

$$g \cdot x = \begin{cases} g(x), & \text{if } g \in G_i; \\ x, & \text{otherwise.} \end{cases}$$

Then we can define a measure-preserving action $G \curvearrowright (\, X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}} \,)$ by

$$g \cdot [(x_i)]_{\mathcal{U}} = [(g \cdot x_i)]_{\mathcal{U}}.$$

It is easily checked that $\mathrm{Fix}_{X_{\mathcal{U}}}(g) = [(\mathrm{Fix}_{\Omega_i}(g))]_{\mathcal{U}}$. Thus $\mathrm{Fix}_{X_{\mathcal{U}}}(g) \in \mathbf{B}^0_{\mathcal{U}}$ and

$$\mu_{\mathcal{U}}(\mathrm{Fix}_{X_{\mathcal{U}}}(g)) = \lim_{\mathcal{U}} |\,\mathrm{Fix}_{\Omega_i}(g)|/|\Omega_i| = \chi(g).$$

Let $f : X_{\mathcal{U}} \to \mathrm{Sub}_G$ be the G-equivariant map defined by $x \mapsto G_x$. Note that for each $g \in G$, we have that

$$f^{-1}(\, \{ H \in \mathrm{Sub}_G \mid g \in H \, \} \,) = \mathrm{Fix}_{X_{\mathcal{U}}}(g) \in \mathbf{B}^0_{\mathcal{U}}.$$

It follows that f is $\mathbf{B}_{\mathcal{U}}$-measurable and hence $\nu = f_* \mu_{\mathcal{U}}$ is an IRS of G. Furthermore, for each $g \in G$,

$$\chi(g) = \mu_{\mathcal{U}}(\mathrm{Fix}_{X_{\mathcal{U}}}(g)) = \nu(\, \{ \, H \in Sub_G \mid g \in H \, \});$$

and so $\chi = \chi_\nu$ is the corresponding associated character. Finally, since χ is a nontrivial indecomposable character, it follows that ν is a nontrivial ergodic IRS; and thus Proposition 4.4 yields that $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth. This completes the proof of Proposition 4.3.

In the proof of Theorem 1.3, we will need to understand the decompositions of arbitary characters χ of full limits with linear natural orbit growth. So suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit with linear natural orbit growth and let $G \curvearrowright (\, \Delta, m \,)$ be the canonical ergodic action. For each $r \geq 1$, let ν_r be the stabilizer distribution of the ergodic action $G \curvearrowright (\, \Delta^r, m^{\otimes r} \,)$. Then, by Proposition 4.3 and Theorem 2.4, the set of indecomposable characters of G is given by

$$\mathcal{E}(G) = \{ \, \chi_{\mathrm{reg}}, \chi_{\mathrm{con}} \, \} \cup \{ \, \chi_{\nu_r} \mid r \in \mathbb{N}^+ \, \}.$$

Proposition 4.9. *With the above hypotheses, for every character $\chi \in \mathcal{F}(G)$, there exist uniquely determined non-negative real coefficients α, β, and γ_r for $r \geq 1$ such that:*

(i) $\alpha + \beta + \sum_{r \geq 1} \gamma_r = 1$*; and*
(ii) $\chi = \alpha \, \chi_{reg} + \beta \, \chi_{con} + \sum_{r \geq 1} \gamma_r \, \chi_{\nu_r}$.

Consequently, $\mu = \alpha \, \delta_1 + \beta \, \delta_G + \sum_{r \geq 1} \gamma_r \, \nu_r$ is the unique *IRS of G such that $\chi_\mu = \chi$.*

Proof. As in the proof of Leinen-Puglisi [7, Theorem 3.6], every convergent sequence of elements of $\mathcal{E}(G)$, which does not tend to one of the functions in

$$\{ \, \chi_{\mathrm{con}} \, \} \cup \{ \, \chi_{\nu_r} \mid r \in \mathbb{N}^+ \, \},$$

must converge to

$$\lim_{r \to \infty} \chi_{\nu_r} = \lim_{r \to \infty} (\chi_{\nu_1})^r = \chi_{\mathrm{reg}},$$

since $\chi_{\nu_1}(g) = m(\mathrm{Fix}_\Delta(g)) < 1$ for all $1 \neq g \in G$. Thus $\mathcal{E}(G)$ is a closed subset of $\mathcal{F}(G)$. By Thoma [13], $\mathcal{F}(G)$ is a Choquet simplex; and, applying Choquet's theorem, we obtain that if $\chi \in \mathcal{F}(G)$, then there exist uniquely determined non-negative real coefficients α, β, and γ_r for $r \geq 1$ such that:

(i) $\alpha + \beta + \sum_{r \geq 1} \gamma_r = 1$; and
(ii) $\chi = \alpha \, \chi_{\mathrm{reg}} + \beta \, \chi_{\mathrm{con}} + \sum_{r \geq 1} \gamma_r \, \chi_{\nu_r}$.

In particular, the IRS $\mu = \alpha\,\delta_1 + \beta\,\delta_G + \sum_{r\geq 1} \gamma_r\,\nu_r$ satisfies $\chi_\mu = \chi$. By considering an element $1 \neq g \in G$ such that $0 < m(\mathrm{Fix}_\Delta(g)) < 1$, we see that if $\nu \neq \nu'$ are two distinct ergodic IRSs of G, then $\chi_\nu \neq \chi_{\nu'}$; and it follows that μ is the unique IRS of G such that $\chi_\mu = \chi$. □

Remark 4.10. It is not true in general that if G is a simple locally finite group and $\nu \neq \nu'$ are two distinct ergodic IRSs of G, then $\chi_\nu \neq \chi_{\nu'}$. For example, let $\mathbb{F} = GF(q)$ be the finite field with q elements and let V be a vector space over $\mathbb{F}$ having a countably infinite basis $\mathcal{B} = \{\, v_1, v_2, \cdots, v_n, \cdots \}$. For each $n \geq 1$, let G_n be the group of linear transformations of V that leave the subspace $V_n = \langle\, v_1, \cdots, v_n \,\rangle$ invariant, induce an element of $SL(V_n)$ on V_n and fix each of the basis vectors in $\mathcal{B} \smallsetminus \{\, v_1, v_2, \cdots, v_n \,\}$. Then the *stable special linear group* $G = \bigcup_{n\geq 1} G_n$ is a simple locally finite group.

Let V^* be the dual space of linear functionals $\varphi : V \to \mathbb{F}_p$ and let λ be the Haar measure on V^*. Then G acts ergodically on $(\, V^*, \lambda\,)$; and, letting ν be the corresponding stabilizer distribution, the associated character is $\chi_\nu(g) = 1/q^{\mathrm{rank}(g-1)}$.

Next let $X = \mathbb{F}^{\mathbb{N}^+}$ and let μ be the uniform product probability measure on X. Then, identifying $\mathbb{F}^n$ with V_n, we can define an ergodic action of G on $(\, X, \mu\,)$ by letting each subgroup G_n act via

$$g \cdot (\, \alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \cdots, \alpha_m, \cdots\,) = (\, g(\alpha_1, \cdots, \alpha_n), \alpha_{n+1}, \cdots, \alpha_m, \cdots).$$

Let ν' be the corresponding stabilizer distribution. Then it is easily checked that the associated character is $\chi_{\nu'}(g) = 1/q^{\mathrm{rank}(g-1)}$ and so $\nu \neq \nu'$ are two distinct ergodic IRSs of G such that $\chi_\nu = \chi_{\nu'}$.

5. The proof of Theorem 1.3

In this section, we will present the proof of Theorem 1.3. Suppose that G is an $L(\mathrm{Alt})$-group and that $G \ncong \mathrm{Alt}(\mathbb{N})$. Then, as explained in Section 1, it is enough to show that every indecomposable character of G is the associated character χ_ν of some ergodic IRS ν of G. First suppose that G has no nontrivial indecomposable characters. Then, since $\chi_{\mathrm{con}} = \chi_{\delta_G}$ and $\chi_{\mathrm{reg}} = \chi_{\delta_1}$, the desired conclusion holds. Hence we can suppose that G has a nontrivial indecomposable character χ. Let $G = \bigcup_{i\in\mathbb{N}} G_i$ be the (not necessarily full) union of the increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. We will begin by expressing $G = \bigcup_{\ell\in\mathbb{N}} G(\ell)$ as a (not necessarily strictly) increasing union of subgroups $G(\ell)$, each of which can be expressed as a full limit of finite alternating groups.

Applying Hall [5, Theorem 5.1], since $G \ncong \mathrm{Alt}(\mathbb{N})$, it follows that for each $i \in \mathbb{N}$, the number c_{ij} of nontrivial G_i-orbits on Δ_j is unbounded as $j \to \infty$. Hence, after passing to a suitable subsequence, we can suppose that each G_i has at least 2 nontrivial orbits on Δ_{i+1}. Since G_i is simple, this implies that if $1 \neq G_i' \leqslant G_i$, then G_i' also has at least 2 nontrivial orbits on Δ_{i+1}. For each $\ell \in \mathbb{N}$, we define sequences of subsets $\Delta_j^\ell \subseteq \Delta_j$ and subgroups $G(\ell)_j = \mathrm{Alt}(\Delta_j^\ell)$ for $j \geq \ell$ inductively as follows:

- $\Delta_\ell^\ell = \Delta_\ell$;
- $\Delta_{j+1}^\ell = \Delta_{j+1} \smallsetminus \mathrm{Fix}_{\Delta_{j+1}}(G(\ell)_j)$.

Clearly each $G(\ell)_j$ is strictly contained in $G(\ell)_{j+1}$ and $G(\ell) = \bigcup_{\ell\leq j\in\mathbb{N}} G(\ell)_j$ is the full limit of the alternating groups $G(\ell)_j = \mathrm{Alt}(\Delta_j^\ell)$. It is also easily checked that if $\ell < m$ and $i < j$, then

$$G_\ell \leqslant G(\ell)_i \leqslant G(m)_i < G(m)_j.$$

It follows that if $\ell < m$, then $G(\ell) \leqslant G(m)$ and that $G = \bigcup_{\ell \in \mathbb{N}} G(\ell)$. For each $\ell \in \mathbb{N}$, let $\chi_\ell = \chi \restriction G(\ell)$.

Lemma 5.1. *The subgroup $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ has linear natural orbit growth for all but finitely many $\ell \in \mathbb{N}$.*

Proof. For the sake of contradiction, suppose that there exists an infinite subset $I \subseteq \mathbb{N}$ such that for all $\ell \in I$, the subgroup $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ does *not* have linear natural orbit growth. Then, by Proposition 4.3, for each $\ell \in I$, the only indecomposable characters of $G(\ell)$ are χ_{con} and χ_{reg}. Hence there exists a real number $0 \leq r_\ell \leq 1$ such that $\chi_\ell = r_\ell\, \chi_{\mathrm{con}} + (1 - r_\ell)\, \chi_{\mathrm{reg}}$. If $\ell < m$ are distinct elements of I, then $G(\ell) \leqslant G(m)$ and it follows that $r_\ell = r_m$. But then there exists a fixed r such that $r_\ell = r$ for all $\ell \in I$ and this implies that $\chi = r\, \chi_{\mathrm{con}} + (1-r)\, \chi_{\mathrm{reg}}$, which is a contradiction. □

Hence we can suppose that $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ has linear natural orbit growth for all $\ell \in \mathbb{N}$. Let $G(\ell) \curvearrowright (\, \Delta_\ell, m_\ell \,)$ be the canonical ergodic action and for each $r \in \mathbb{N}^+$, let $\nu(\ell)_r$ be the stabilizer distribution of $G(\ell) \curvearrowright (\, \Delta_\ell^r, m_\ell^{\otimes r} \,)$. Then for each $\ell \in \mathbb{N}$, there exist $\alpha(\ell)$, $\beta(\ell)$, $\gamma(\ell)_r \in [\,0, 1\,]$ with $\alpha(\ell) + \beta(\ell) + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r = 1$ such that

$$\chi_\ell = \alpha(\ell)\, \chi_{\mathrm{reg}} + \beta(\ell)\, \chi_{\mathrm{con}} + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r\, \chi_{\nu(\ell)_r}. \tag{5.1}$$

Thus χ_ℓ is the associated character χ_{ν_ℓ} of the IRS ν_ℓ of $G(\ell)$ defined by

$$\nu_\ell = \alpha(\ell)\, \delta_1 + \beta(\ell)\, \delta_{G(\ell)} + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r\, \nu(\ell)_r. \tag{5.2}$$

For each $\ell \in \mathbb{N}$, let $f_\ell : \mathrm{Sub}_{G(\ell+1)} \to \mathrm{Sub}_{G(\ell)}$ be the continuous map defined by $H \mapsto H \cap G(\ell)$.

Lemma 5.2. *$(f_\ell)_* \nu_{\ell+1} = \nu_\ell$.*

Proof. Let θ_ℓ be the character associated with the IRS $(f_\ell)_* \nu_{\ell+1}$ of $G(\ell)$. Then for each element $g \in G(\ell)$,

$$\begin{aligned}
\theta_\ell(g) &= (f_\ell)_* \nu_{\ell+1}(\, \{\, K \in \mathrm{Sub}_{G(\ell)} \mid g \in K \,\}\,) \\
&= \nu_{\ell+1}(\, \{\, H \in \mathrm{Sub}_{G(\ell+1)} \mid g \in H \,\}\,) \\
&= \chi_{\ell+1}(g) \\
&= \chi_\ell(g).
\end{aligned}$$

Hence the result follows from Proposition 4.9. □

Thus $\{\, (\, \mathrm{Sub}_{G(\ell)}, \nu_\ell \,) \mid \ell \in \mathbb{N} \,\}$ is an inverse family of topological probability spaces in the sense of Choksi [2]; and clearly we can naturally identify the inverse limit $\varprojlim \mathrm{Sub}_{G(\ell)}$ with Sub_G. For each $\ell \in \mathbb{N}$, let $f_{\infty\ell} : \mathrm{Sub}_G \to \mathrm{Sub}_{G(\ell)}$ be the continuous map defined by $H \mapsto H \cap G(\ell)$. Applying Choksi [2, Theorem 2.2], since each $\mathrm{Sub}_{G(\ell)}$ is a compact Hausdorff space, it follows that there exists a measure

ν on Sub_G such that $(f_{\infty\ell})_*\nu = \nu_\ell$ for each $\ell \in \mathbb{N}$. Note that for each $\ell \in \mathbb{N}$ and element $g \in G(\ell)$, we have that

$$\begin{aligned}\chi(g) &= \nu_\ell(\{ K \in \mathrm{Sub}_{G(\ell)} \mid g \in K \}) \\ &= (f_{\infty\ell})_*\nu(\{ K \in \mathrm{Sub}_{G(\ell)} \mid g \in K \}) \\ &= \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \}).\end{aligned}$$

Thus χ is the character associated with the IRS ν of G; and since χ is a nontrivial indecomposable character, it follows that ν is a nontrivial ergodic IRS. This completes the proof of Theorem 1.3.

6. The indecomposable characters of $\mathrm{Alt}(\mathbb{N})$

In this final section, we will point out the two ways in which Theorem 1.3 fails when $G = \mathrm{Alt}(\mathbb{N})$. Firstly, it follows from Thomas-Tucker-Drob [15, Theorem 9.2] that there exist ergodic IRSs ν of $\mathrm{Alt}(\mathbb{N})$ such that the associated character

$$\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \})$$

is *not* indecomposable. Secondly, as we will explain in the remainder of this section, there exist indecomposable characters χ of $\mathrm{Alt}(\mathbb{N})$ for which there does *not* exist an ergodic IRS ν such that $\chi = \chi_\nu$.

We will begin by recalling Thoma's classification [14] of the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$. For each $g \in \mathrm{Alt}(\mathbb{N})$ and $n \geq 2$, let $c_n(g)$ be the number of cycles of length n in the cyclic decomposition of the permutation g. Then the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$ are precisely the functions $\chi : \mathrm{Alt}(\mathbb{N}) \to \mathbb{C}$ such that there exist two sequences $(\alpha_i \mid i \in \mathbb{N}^+)$ and $(\beta_i \mid i \in \mathbb{N}^+)$ of non-negative real numbers satisfying

- $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \geq 0$;
- $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_i \geq \cdots \geq 0$;
- $\sum_{i=1}^\infty \alpha_i + \sum_{i=1}^\infty \beta_i \leq 1$;

and such that for all $g \in \mathrm{Alt}(\mathbb{N})$,

$$\chi(g) = \prod_{n=2}^\infty s_n^{c_n(g)}, \qquad \text{where } s_n = \sum_{i=1}^\infty \alpha_i^n + (-1)^{n+1} \sum_{i=1}^\infty \beta_i^n.$$

(In these products, s_n^0 is always taken to be 1, including the case when $s_n = 0$.)

Proposition 6.1. *If χ is the indecomposable character for which $\alpha_1 = \beta_1 = 1/2$ and $\alpha_i = \beta_i = 0$ for all $i > 1$, then there does not exist an ergodic IRS ν of* $\mathrm{Alt}(\mathbb{N})$ *such that $\chi = \chi_\nu$.*

Proof. Suppose that ν is an ergodic IRS of $\mathrm{Alt}(\mathbb{N})$ such that $\chi = \chi_\nu$. Note that

$$s_n = \begin{cases} (1/2)^{n-1}, & \text{if } n > 1 \text{ is odd;} \\ 0, & \text{if } n > 1 \text{ is even;} \end{cases}$$

and hence if $g \in \mathrm{Alt}(\mathbb{N})$, then $\chi(g) = 0$ if and only if $c_n(g) \geq 1$ for some even integer $n > 1$. It follows that ν-a.e. $H \in \mathrm{Sub}_G$ contains an element consisting of a single 3-cycle, but does not contain any elements consisting of a product of two 2-cycles. However, Thomas-Tucker-Drob [15, Theorem 9.1] implies that if ν is an ergodic IRS of $\mathrm{Alt}(\mathbb{N})$ such that $\nu \neq \delta_1$, then for ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})}$, there exists an infinite subset $B \subseteq \mathbb{N}$ such that $\mathrm{Alt}(B) \leqslant H$. □

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