PRACTICE SECOND MID-TERM QUESTIONS

Question 1.  (a) Give the definition of $\Gamma \models \varphi$ and $\Gamma \vdash \varphi$.
(b) State the Soundness Theorem.
(c) Let $\mathcal{L}$ have the following nonlogical symbols:
   i) a binary predicate symbol $<$; and
   iii) a unary function symbol $f$.
Let $T$ be the theory in $\mathcal{L}$ with the following axioms:
(1) $(\forall x)\neg(x < x)$
(2) $(\forall x)(\forall y)(x < y \lor y < x \lor x = y)$
(3) $(\forall x)(\forall y)(\forall z)([x < y \land y < z] \rightarrow [x < z])$
(4) $(\forall x)(\forall y)([x < y] \rightarrow (\exists z)[x < z \land z < y])$
(5) $(\forall x)(\exists y)(\exists z)(y < x \land x < z)$
(6) $(\forall x)(\forall y)([x < y] \rightarrow [fx < fy])$.
(7) $(\forall x)(x < fx)$.
Let $\sigma$ be the sentence

$$(\forall x)(\exists y)(fy = x).$$

Prove that $T \not\models \sigma$ and that $T \not\models \neg \sigma$.

Question 2.  (a) State the Soundness Theorem and the Completeness
     Theorem for predicate logic.
(b) State and prove the Los-Vaught Theorem.
(c) Let $\mathcal{L}$ have the following nonlogical symbols:
   • a binary predicate symbol $E$; and
   • two unary predicate symbols $P$ and $Q$.
For each $n \geq 1$, let $\varphi_n$ be the sentence which says that there exist at
least $n$ distinct elements satisfying the predicate $P$. For example, $\varphi_2$ is the
sentence:

$$(\exists x)(\exists y)(x \neq y \land Px \land Py).$$
Similarly, for each $n \geq 1$, let $\psi_n$ be the sentence which says that there exist at least $n$ distinct elements satisfying the predicate $Q$. Let $T$ be the theory in $\mathcal{L}$ with the following axioms:

- $(\forall x)(Px \vee Qx)$
- $(\forall x)\neg(Px \land Qx)$
- $\varphi_n$ for all $n \geq 1$.
- $\psi_n$ for all $n \geq 1$.
- $(\forall x)(\forall y)( Exy \leftrightarrow (Px \land Qy ))$

Prove that $T$ is consistent and complete.