PRACTICE FINAL EXAM

1. Set Theory

Question 1.

(a) State and prove the Cantor-Bernstein Theorem.

(b) Let $\mathbb{Z}[x]$ be the set of polynomials with integer coefficients in the variable x; i.e. each $p(x) \in \mathbb{Z}[x]$ has the form

$$p(x) = z_0 + z_1 x + \cdots + z_n x^n,$$

where $n \geq 0$ and $z_0, \ldots, z_n \in \mathbb{Z}$. Prove that $\mathbb{N} \sim \mathbb{Z}[x]$.

(c) A function $f : \mathbb{N} \to \mathbb{N}$ is said to be a *quasi-identity function* iff

$$|f(n) - n| \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Determine whether the set $QI(\mathbb{N})$ of quasi-identity functions is countable or uncountable.

Question 2.

(a) Give the definition of a linear ordering.

(b) Give the definition of an isomorphism between two linear orders $\langle L_1, < \rangle$ and $\langle L_2, \prec \rangle$.

Prove or disprove each of the following statements.

(c) $\langle \mathbb{Q} \smallsetminus \{0, 1\}, < \rangle \simeq \langle \mathbb{Q} \smallsetminus [0, 1], < \rangle$.

(d) $\langle \mathbb{Q} \smallsetminus \{0, 1\}, < \rangle \simeq \langle \mathbb{Q} \smallsetminus (0, 1), < \rangle$.

(e) $\langle \mathbb{Q} \smallsetminus \mathbb{Z}, < \rangle \simeq \langle \mathbb{R} \smallsetminus \mathbb{Z}, < \rangle$.

2. Propositional Logic

Question 3.

(a) Give the definition of $\Sigma \models \alpha$, where Σ is a set of propositional wffs and α is a propositional wff.

(b) Which of the following wffs are tautologies?

(i) $(Q \vee (\neg(P \to Q)))$.

(ii) $((P \to (Q \to R)) \leftrightarrow ((P \wedge Q) \to R))$.

(c) A set Σ of wffs is said to be *independent* iff for all $\varphi \in \Sigma$, $\Sigma \setminus \{\varphi\} \not\models \varphi$. Prove that if Σ is a finite set of wffs, then there exists a subset $\Delta \subseteq \Sigma$ such that:

(i) Δ is independent; and

(ii) for all $\varphi \in \Sigma$, $\Delta \models \varphi$.

(*Hint:* Consider a subset $\Delta \subseteq \Sigma$ of *minimal* size such that for all $\varphi \in \Sigma$, $\Delta \models \varphi$.)

Question 4.

(a) State the Compactness Theorem for propositional logic.

(b) Let $\{S_n \mid n \in \mathbb{N}\}$ be a collection of finite subsets of $\mathbb{N}$ such that for each finite subset $F \subset \mathbb{N}$, there exists a subset $A_F \subseteq \mathbb{N}$ with $|A_F \cap S_n| = 1$ for all $n \in F$. Prove that there exists a subset $A \subseteq \mathbb{N}$ such that $|A \cap S_n| = 1$ for all $n \in \mathbb{N}$.

(c) Give an example of a collection $\{T_n \mid n \in \mathbb{N}\}$ of subsets of $\mathbb{N}$ satisfying both of the following conditions.

(i) For each finite subset $F \subset \mathbb{N}$, there exists a subset $B_F \subseteq \mathbb{N}$ such that $|B_F \cap T_n| = 1$ for all $n \in F$.

(ii) There does *not* exist a subset $B \subseteq \mathbb{N}$ such that $|B \cap T_n| = 1$ for all $n \in \mathbb{N}$.

3. Predicate Logic

Question 5.

(a) Give the definition of $\Gamma \models \varphi$ and $\Gamma \vdash \varphi$.

(b) State the Completeness Theorem and the Soundness Theorem for predicate logic.

(c) Let $\mathcal{L}$ have the following nonlogical symbols :

(i) a binary predicate symbol $<$; and

(ii) two unary predicate symbols P and Q.

Let T be the theory in $\mathcal{L}$ with the following axioms:

(1) $(\forall x)\neg(x < x)$

(2) $(\forall x)(\forall y)(x < y \vee y < x \vee x = y)$

(3) $(\forall x)(\forall y)(\forall z)([x < y \wedge y < z] \rightarrow [x < z])$

(4) $(\forall x)(\forall y)([x < y] \rightarrow (\exists z)[x < z \wedge z < y])$

(5) $(\forall x)(\exists y)(\exists z)(y < x \wedge x < z)$

(6) $(\forall x)(Px \vee Qx)$

(7) $(\forall x)\neg(Px \wedge Qx)$

(8) $(\forall x)(\exists y)(\exists z)(y < x \wedge Py \wedge x < z \wedge Pz)$

(9) $(\forall x)(\exists y)(\exists z)(y < x \wedge Qy \wedge x < z \wedge Qz)$

Let σ be the following sentence:

$$(\forall x)(\forall y)([x < y \wedge Px \wedge Py] \rightarrow (\exists z)[x < z \wedge z < y \wedge Pz]).$$

Prove that $T \nvdash \sigma$ and $T \nvdash \neg\sigma$.

Question 6. Recall that two structures $\mathcal{A}$, $\mathcal{B}$ for the first-order language $\mathcal{L}$ are said to be *elementarily equivalent*, written $\mathcal{A} \equiv \mathcal{B}$, if for every sentence σ, we have that $\mathcal{A} \vDash \sigma$ iff $\mathcal{B} \vDash \sigma$.

(a) Let $\mathbf{C}$ be an axiomatizable class of structures for the first-order language $\mathcal{L}$ and let $\mathcal{A}$, $\mathcal{B}$ be any structures for the language $\mathcal{L}$. Prove that if $\mathcal{A} \equiv \mathcal{B}$, then $\mathcal{A} \in \mathbf{C}$ iff $\mathcal{B} \in \mathbf{C}$.

(b) State the Compactness Theorem for predicate logic.

(c) A graph Γ is *connected* if for every pair of distinct vertices c, $d \in V$, there exists a path in Γ from c to d. Otherwise, Γ is *nonconnected*. For example, if $\Gamma = \langle \mathbb{Z}, E \rangle$ is the graph such that nEm iff $|n - m| = 1$, then Γ is connected. Prove that there exists a graph G such that $G \equiv \Gamma$ and G is nonconnected.

(d) Prove that the class of nonconnected graphs is not axiomatizable.