1. Set Theory

Question 1. (a) State and prove the Cantor-Bernstein Theorem.

(b) Let $\mathbb{Z}[x]$ be the set of polynomials with integer coefficients in the variable $x$; i.e. each $p(x) \in \mathbb{Z}[x]$ has the form
\[ p(x) = z_0 + z_1x + \cdots + z_nx^n, \]
where $n \geq 0$ and $z_0, \ldots, z_n \in \mathbb{Z}$. Prove that $\mathbb{N} \sim \mathbb{Z}[x]$.

(c) A function $f : \mathbb{N} \to \mathbb{N}$ is said to be a quasi-identity function iff
\[ |f(n) - n| \leq 1 \quad \text{for all } n \in \mathbb{N}. \]
Determine whether the set $QI(\mathbb{N})$ of quasi-identity functions is countable or uncountable.

Question 2. (a) Give the definition of a linear ordering.

(b) Give the definition of an isomorphism between two linear orders $\langle L_1, < \rangle$ and $\langle L_2, \prec \rangle$.

Prove or disprove each of the following statements.

(c) $\langle \mathbb{Q} \setminus \{0, 1\}, < \rangle \simeq \langle \mathbb{Q} \setminus [0, 1], < \rangle$.

(d) $\langle \mathbb{Q} \setminus \{0, 1\}, < \rangle \simeq \langle \mathbb{Q} \setminus (0, 1), < \rangle$.

(e) $\langle \mathbb{Q} \setminus \mathbb{Z}, < \rangle \simeq \langle \mathbb{R} \setminus \mathbb{Z}, < \rangle$.

2. Propositional Logic

Question 3. (a) Give the definition of $\Sigma \models \alpha$, where $\Sigma$ is a set of propositional wffs and $\alpha$ is a propositional wff.

(b) Which of the following wffs are tautologies?

(i) $(Q \lor (\neg(P \to Q)))$.

(ii) $((P \to (Q \to R)) \leftrightarrow ((P \land Q) \to R))$. 
2 ANSWER ALL OF THE QUESTIONS.

(c) Suppose that $\alpha$ is a wff which only involves connectives taken from the set \{\lor, \land, \neg\}. Let $\alpha^*$ the wff obtained by

- replacing each $\lor$ of $\alpha$ by $\land$,
- replacing each $\land$ of $\alpha$ by $\lor$, and
- replacing each sentence symbol $A$ of $\alpha$ by its negation ($\neg A$).

Prove by induction on the length of $\alpha$ that $(\alpha \leftrightarrow (\neg \alpha^*))$ is a tautology.

**Question 4.**

(a) State the Compactness Theorem for propositional logic.

Suppose that $\Gamma = \langle V, E \rangle$ is a graph, where $V$ is the set of vertices and $E \subseteq V \times V$ is the edge relation. Then $\Gamma$ is said to be bipartite iff it is possible to partition the vertices $V = P \sqcup Q$ so that every edge joins some vertex in $P$ to some vertex in $Q$. (In other words, no pair of vertices in $P$ are adjacent and no pair of vertices in $Q$ are adjacent.)

(b) Prove that if $\Gamma$ is a countable graph such that every finite subgraph $\Gamma_0$ is bipartite, then $\Gamma$ is bipartite.

3. **Predicate Logic**

**Question 5.**

(a) Give the definition of $\Gamma \models \varphi$ and $\Gamma \vdash \varphi$.

(b) State the Completeness Theorem and the Soundness Theorem for predicate logic.

(c) Let $\mathcal{L}$ have the following nonlogical symbols:

(i) a binary predicate symbol $<$; and

(ii) two unary predicate symbols $P$ and $Q$.

Let $T$ be the theory in $\mathcal{L}$ with the following axioms:

1. $(\forall x) \neg (x < x)$
2. $(\forall x)(\forall y)(x < y \lor y < x \lor x = y)$
3. $(\forall x)(\forall y)(\forall z)([x < y \land y < z] \rightarrow [x < z])$
4. $(\forall x)(\forall y)([x < y] \rightarrow (\exists z)[x < z \land z < y])$
5. $(\forall x)(\exists y)(\exists z)(y < x \land x < z)$
6. $(\forall x)(P x \lor Q x)$
7. $(\forall x)\neg(P x \land Q x)$
8. $(\forall x)(\exists y)(\exists z)(y < x \land P y \land x < z \land P z)$
9. $(\forall x)(\exists y)(\exists z)(y < x \land Q y \land x < z \land Q z)$
Let $\sigma$ be the following sentence:

$$(\forall x)(\forall y)([x < y \land Px \land Py] \rightarrow (\exists z)[x < z \land z < y \land Pz]).$$

Prove that $T \not\vdash \sigma$ and $T \not\vdash \neg \sigma$.

(d) Let $L$, $T$ and $\sigma$ be as in part (c). Let $\varphi$ be the following sentence:

$$(\exists x)(\exists y)(x < y \land (\forall z)[x < z \land z < y \rightarrow Qz]).$$

Prove that $T \vdash (\neg \sigma \rightarrow \varphi)$.

Question 6. (a) Let $C$ be an axiomatizable class of structures for the first-order language $L$; and let $A$, $B$ be any structures for the language $L$. Prove that if $A \equiv B$, then $A \in C$ iff $B \in C$.

(b) State the Compactness Theorem for predicate logic.

(c) A partial order $P = \langle A, \prec \rangle$ is well-founded iff there does not exist an infinite descending sequence

$$a_1 > a_2 > a_3 > \cdots > a_n > \cdots$$

of elements of $A$. For example, let $D = \langle \mathbb{N}^+, \prec \rangle$ be the partial order on the set $\mathbb{N}^+ = \{ n \in \mathbb{N} \mid n \geq 1 \}$ defined by

$$a \prec b \iff \text{there exists an integer } c > 1 \text{ such that } b = ac.$$

Then $D$ is well-founded. Prove that there exists a partial order $P$ such that $P \equiv D$ and $P$ is not well-founded.

(d) Prove that the class of well-founded partial orders is not axiomatizable.