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<table>
<thead>
<tr>
<th>Question</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
</tbody>
</table>
Question 1. Throughout this question, let $p$ be a prime.

(a) Suppose that $G$ is a finite $p$-group and that $X$ is a nonempty $G$-set such that $|X| \not\equiv 0 \mod p$. Prove that there exists a point $x \in X$ such that $g \cdot x = x$ for all $g \in G$.

(b) Suppose that $G$ is a $p$-group and that $H \trianglelefteq G$. Prove that if $H \neq \{1\}$, then $H \cap Z(G) \neq \{1\}$.

(Hint: Consider the action of $G$ on $H \setminus \{1\}$ by conjugation.)

(a) Let $|G| = p^n$. Let $X = \Omega_1 \sqcup \cdots \sqcup \Omega_t$ be the decomposition of $X$ into $G$-orbits. If $\alpha \in \Omega_i$, then $|\Omega_i| = [G : G\alpha] = p^{m_i}$ for some $0 \leq m_i \leq n$. Since $|X| = \sum_{i=1}^t |\Omega_i|$ and $p$ does not divide $|X|$, there exists $1 \leq i_0 \leq t$ such that $m_{i_0} = 0$. Hence, letting $\Omega_{i_0} = \{x\}$, it follows that $g \cdot x = x$ for all $g \in G$.

(b) Let $|G| = p^n$. If $g \in G$, then $gHg^{-1} = H$ and so $g(H \setminus \{1\})g^{-1} = H \setminus \{1\}$. Thus $G$ acts by conjugation on $H \setminus \{1\}$. Since $|H| = p^m$ for some $1 \leq m \leq n$, it follows that $p$ does not divide $|H \setminus \{1\}|$. Hence there exists $h \in H \setminus \{1\}$ such that $ghg^{-1} = h$ for all $g \in G$. Clearly $h \in H \cap Z(G)$. 
Question 2.  

(a) State the Third Sylow Theorem.
(b) Prove that there does not exist a simple group of order 5500.
(c) Give an example of a nonabelian group of order 5500.

(a) Suppose that $G$ is a finite group of order $n = p^e m$, where $p$ is a prime, $e \geq 1$ and $p$ does not divide $m$. If $s$ is the number of Sylow $p$-subgroups of $G$, then $s$ divides $m$ and $s \equiv 1 \mod p$.

(b) Suppose $G$ is a simple group of order $5^3 \times 11 \times 2^2$. If $s$ is the number of Sylow 5-subgroups of $G$, then $s$ divides 44 and $s \equiv 1 \mod 5$. Since $G$ is simple, $s \neq 1$ and so $s = 11$. By considering the transitive action of $G$ by conjugation on the set of its Sylow 5-subgroups, we see that there is an embedding of $G$ into $S_{11}$. But this is impossible, since $5^3$ does not divide $|S_{11}|$.

(c) Since $|\text{Aut}(C_{11})| = 10$, there exist embeddings

$$C_2 \hookrightarrow \text{Aut}(C_{11}) \quad \text{and} \quad C_5 \hookrightarrow \text{Aut}(C_{11}),$$

which give rise to corresponding nonabelian semidirect products. Thus the nonabelian groups of order 5500 include:

- $(C_{11} \rtimes C_2) \times C_{250}$
- $(C_{11} \rtimes C_5) \times C_{100}$
- etc.
**Question 3.** Suppose that $G$ be a simple group of order 168. Let $P$ be a Sylow 7-subgroup of $G$ and let $H = N_G(P)$.

(a) Prove that $|H| = 21$.

(b) Prove that $N_G(H) = H$. (Hint: Notice that $H \leq N_G(H) \leq G$.)

(c) Prove that there exists an element $g \in G$ such that $gHg^{-1} \neq H$ and $gHg^{-1} \cap H \neq \{1\}$.

(a) If $s$ is the number of Sylow 7-subgroups of $G$, then $s$ divides 24 and $s \equiv 1\mod 7$. Since $G$ is simple, $s \neq 1$ and so $s = 8$. By considering the transitive action of $G$ by conjugation on the set of its Sylow 7-subgroups, we see that $[G : N_G(P)] = 8$ and hence $|H| = |N_G(P)| = 21$.

(b) Since $H \leq N_G(H) \leq G$, it follows that $d = [G : N_G(H)]$ divides $[G : H] = 8$. Also by considering the transitive action of $G$ on the coset space $G/N_G(H)$, we see that there is an embedding of $G$ into $S_d$. Thus 7 divides $|S_d|$ and so $d = 8$. It follows that $N_G(H) = H$.

(c) Suppose that $gHg^{-1} \cap H = 1$ whenever $gHg^{-1} \neq H$. Then the 8 distinct conjugates of $H$ intersect pairwise in 1. Hence

$$| \left( \bigcup_{g \in G} H^g \right) \setminus \{1\}| = 8 \times 20 = 160.$$  

But this means that $G$ has a unique Sylow 2-subgroup, which is a contradiction.
Question 4. Prove that \( (x, y \mid x^2 = 1, y^2 = 1, (xy)^3 = 1) \) is a presentation of \( S_3 \).

Let \( X = \{x, y\} \) and let \( N \) be the normal closure of \( \{x^2, y^2, (xy)^3\} \) in \( F(X) \). For each \( w \in F(X) \), let \( \bar{w} = wN \in F(X)/N \). By von Dyck’s Theorem, there exists a surjective homomorphism \( \varphi : F(X)/N \to S_3 \) such that \( \varphi(\bar{x}) = (1 2) \) and \( \varphi(\bar{y}) = (2 3) \). In particular, \( |F(X)/N| \geq 6 \). On the other hand, let

\[
\bar{w} = \bar{x}^{n_1} \bar{y}^{m_1} \cdots \bar{x}^{n_t} \bar{y}^{m_t} \in F(X)/N,
\]

where each \( n_i, m_i \in \mathbb{Z} \). Since \( \bar{x}^2 = 1 \) and \( \bar{y}^2 = 1 \), we can suppose that each \( 0 \leq n_i, m_i \leq 1 \). Using the relations \( \bar{x}\bar{y}\bar{x}\bar{y} = 1 \) and \( \bar{x} = \bar{x}^{-1} \) and \( \bar{y} = \bar{y}^{-1} \), we can now reduce \( \bar{w} \) to one of the following words:

\[
1, \bar{x}, \bar{y}, \bar{x}\bar{y}, \bar{y}\bar{x}, \bar{x}\bar{y}\bar{x}.
\]

Thus \( |F(X)/N| \leq 6 \) and it follows that \( \varphi : F(X)/N \to S_3 \) is an isomorphism.
Question 5. Recall that if \( \pi \in \text{Sym}(X) \), then \( \text{supp}(\pi) = \{ x \in X \mid \pi(x) \neq x \} \). Let \( S_\infty \) and \( A_\infty \) be the subgroups of \( \text{Sym}({\mathbb{N}}^+) \) defined by

- \( S_\infty = \{ \pi \in \text{Sym}({\mathbb{N}}^+) : |\text{supp}(\pi)| < \infty \} \)
- \( A_\infty = \{ \pi \in \text{Sym}({\mathbb{N}}^+) : |\text{supp}(\pi)| < \infty \text{ and } \pi \text{ is an even permutation}\} \).

Prove that \( A_\infty \) is the unique nontrivial proper normal subgroup of \( S_\infty \).

For each \( n \geq 1 \), define

\[ G_n = \{ \pi \in S_\infty \mid \text{supp}(\pi) \subseteq \{1, \cdots, n\} \} \]

and

\[ H_n = \{ \pi \in A_\infty \mid \text{supp}(\pi) \subseteq \{1, \cdots, n\} \} \].

Then we have that

- \( G_n \cong S_n \) and \( H_n \cong A_n \).
- \( S_\infty = \bigcup_{n \geq 1} G_n \) and \( A_\infty = \bigcup_{n \geq 1} H_n \).

Suppose that \( N \) is a nontrivial proper normal subgroup of \( S_\infty \) and let \( 1 \neq \pi \in N \). Then there exists \( n_0 \geq 5 \) such that \( \pi \in G_{n_0} \). It follows that \( N \cap G_n \) is a nontrivial normal subgroup of \( G_n \) for each \( n \geq n_0 \); and this implies that either \( N \cap G_n = H_n \) or \( N \cap G_n = G_n \). In particular, \( H_n \leq N \cap G_n \) and so

\[ A_\infty = \bigcup_{n \geq n_0} H_n \leq N. \]

It is easily checked that \( |S_\infty : A_\infty| = 2 \). Hence, since \( N \) is a proper subgroup of \( S_\infty \), it follows that \( N = A_\infty \).