361 SECOND MIDTERM 2017

Question 1.

(a) Give the definition of $A \approx B$; i.e. the set A is equinumerous to the set B.

(b) Prove that if A is any set, then $A \not\approx \mathcal{P}(A)$.

Question 2.

(a) State the Cantor-Bernstein Theorem.

(b) Prove that $\mathbb{Z} \times \mathbb{Q} \sim \mathbb{N}$.

(c) Let $S(\mathbb{N})$ be the set of *strictly increasing* functions $f : \mathbb{N} \to \mathbb{N}$;
ie. those functions f such that $f(n) < f(m)$ for all $0 \leq n < m$.
Prove that $\mathcal{P}(\mathbb{N}) \approx S(\mathbb{N})$.

Question 3.

(a) Define the addition operation $\kappa + \lambda$ for cardinal numbers κ and λ.

(b) Prove that $\aleph_0 + 2^{\aleph_0} = 2^{\aleph_0}$.

Question 4.

(a) Give the definition of a well-ordering $<$ of a set W.

(b) Let $\prec$ be the linear ordering on $\mathbb{N} \times \mathbb{N}$ defined by $\langle a, b \rangle \prec \langle c, d \rangle$ if either:

- $a < c$, or
- $a = c$ and $b < d$.

Prove that $\prec$ is a well-ordering of $\mathbb{N} \times \mathbb{N}$.

[*Note:* You do not need to prove that $\prec$ is a linear ordering of $\mathbb{N} \times \mathbb{N}$.]

(c) Prove that $\langle\, \mathbb{N} \times \mathbb{N}\,, \prec\, \rangle$ is *not* isomorphic to $\langle\, \mathbb{N}\,, <\, \rangle$.