10 Construction of $\mathbb{Q}$

Most proofs are left as reading exercises.

Definition 10.1. $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$.

Definition 10.2. Let $\sim$ be the binary relation defined on $\mathbb{Z} \times \mathbb{Z}'$ by

$$\langle a, b \rangle \sim \langle c, d \rangle \quad \text{iff} \quad ad = cb.$$

Theorem 10.3. *$\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}'$.*

Proof. We just check that $\sim$ is transitive. So suppose that $\langle a, b \rangle \sim \langle c, d \rangle$ and $\langle c, d \rangle \sim \langle e, f \rangle$. Then

$$\begin{aligned} ad &= cb \quad (1) \\ cf &= ed \quad (2) \end{aligned}$$

Multiplying (1) by f and (2) by b, we obtain

$$\begin{aligned} adf &= cbf \quad (3) \\ cfb &= edb \quad (4) \end{aligned}$$

Hence $adf = edb$. Since $d \neq 0$, the Cancellation Law implies that $af = eb$. Hence $\langle a, b \rangle \sim \langle e, f \rangle$. □

Definition 10.4. The set $\mathbb{Q}$ of *rational numbers* is defined by

$$\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}' / \sim$$

ie. $\mathbb{Q}$ is the set of $\sim$–equivalence classes.

Notation For each $\langle a, b \rangle \in \mathbb{Z} \times \mathbb{Z}'$, the corresponding equivalence class is denoted by $[\langle a, b \rangle]$.

Next we want to define an addition opperation on $\mathbb{Q}$. [Note that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}.$$

This suggests we make the following definition.]

Definition 10.5. We define the binary operation $+_{\mathbb{Q}}$ on $\mathbb{Q}$ by

$$[\langle a, b \rangle] +_{\mathbb{Q}} [\langle c, d \rangle] = [\langle ad + cd, bd \rangle].$$

Remark 10.6. Since $b \neq 0$ and $d \neq 0$ we have that $bd \neq 0$ and so $\langle ad + cb, bd \rangle \in \mathbb{Z} \times \mathbb{Z}'$.

Lemma 10.7. $+_{\mathbb{Q}}$ *is well-defined.* $\square$

Theorem 10.8. *For all $q, r, s \in \mathbb{Q}$, we have that*

$$q +_{\mathbb{Q}} r = r +_{\mathbb{Q}} q$$

$$q +_{\mathbb{Q}} (r +_{\mathbb{Q}} s) = (q +_{\mathbb{Q}} r) +_{\mathbb{Q}} s.$$

Definition 10.9 (Identity element for $+_{\mathbb{Q}}$). $0_{\mathbb{Q}} = [\langle 0, 1\rangle]$.

Theorem 10.10.

(a) For all $q \in \mathbb{Q}$, $q +_{\mathbb{Q}} 0_{\mathbb{Q}} = q$.

(b) For any $q \in \mathbb{Q}$, there exists a unique $r \in \mathbb{Q}$ such that $q +_{\mathbb{Q}} r = 0_{\mathbb{Q}}$.

Proof. (a) Let $q = [\langle a, b\rangle]$. Then

$$\begin{aligned} q +_{\mathbb{Q}} 0_{\mathbb{Q}} &= [\langle a, b\rangle] +_{\mathbb{Q}} [\langle 0, 1\rangle] \\ &= [\langle a \cdot 1 + 0 \cdot b, b \cdot 1\rangle] \\ &= [\langle a, b\rangle] \\ &= q. \end{aligned}$$

To show that there exists at least one such element, consider $r = [\langle -a, b\rangle]$. Then

$$\begin{aligned} q +_{\mathbb{Q}} r &= [\langle a, b\rangle] +_{\mathbb{Q}} [\langle -a, b\rangle] \\ &= [\langle ab + (-a)b, b^2\rangle] \\ &= [\langle 0, b^2\rangle] \end{aligned}$$

Since $0 \cdot 1 = 0 \cdot b^2$, we have $\langle 0, b^2\rangle = \langle 0, 1\rangle$. Hence

$$\begin{aligned} q +_{\mathbb{Q}} r &= [\langle 0, b^2\rangle] \\ &= [\langle 0, 1\rangle] \\ &= 0_{\mathbb{Q}} \end{aligned}$$

As before, simple algebra shows that there exists at most one such element. $\square$

Definition 10.11. For any $q \in \mathbb{Q}$, $-q$ is the unique element of $\mathbb{Q}$ such that

$$q +_{\mathbb{Q}} (-q) = 0_{\mathbb{Q}}.$$

Definition 10.12. We define the binary operation $-_{\mathbb{Q}}$ on $\mathbb{Q}$ by

$$q -_{\mathbb{Q}} r = q +_{\mathbb{Q}} (-r).$$

Next we want to define a multiplication operation on $\mathbb{Q}$. [Note that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

This suggests we make the following definition.]

Definition 10.13. We define the binary operation $\cdot_{\mathbb{Q}}$ on $\mathbb{Q}$ by

$$[\langle a, b\rangle] \cdot_{\mathbb{Q}} [\langle c, d\rangle] = [\langle ac, bd\rangle].$$

Remark 10.14. Since $b \neq 0$ and $d \neq 0$m wee have that $bd \neq 0$ and so $\langle ac, bd\rangle \in \mathbb{Z}\times\mathbb{Z}'$.

Lemma 10.15. *$\cdot_{\mathbb{Z}}$ is well-defined.*

Theorem 10.16. *For all $q, r, s \in \mathbb{Q}$, we have that*

$$\begin{aligned} q \cdot_{\mathbb{Q}} r &= r \cdot_{\mathbb{Q}} q \\ (q \cdot_{\mathbb{Q}} r) \cdot_{\mathbb{Q}} s &= q \cdot_{\mathbb{Q}} (r \cdot_{\mathbb{Q}} s) \\ q \cdot_{\mathbb{Q}} (r +_{\mathbb{Q}} s) &= (q \cdot_{\mathbb{Q}} r) +_{\mathbb{Q}} (q \cdot_{\mathbb{Q}} s) \end{aligned}$$

□

Definition 10.17 (Identity element for $\cdot_{\mathbb{Q}}$).

$$1_{\mathbb{Q}} = [\langle 1, 1\rangle].$$

Theorem 10.18.

(a) For all $q \in \mathbb{Q}$, $q \cdot_{\mathbb{Q}} 1_{\mathbb{Q}} = q$.

(b) For every $0_{\mathbb{Q}} \neq q \in \mathbb{Q}$, there exists a unique $r \in \mathbb{Q}$ such that $q \cdot_{\mathbb{Q}} r = 1_{\mathbb{Q}}$.

Proof. (a) Let $q = [\langle a, b\rangle]$. Then

$$\begin{aligned} q \cdot_{\mathbb{Q}} 1_{\mathbb{Q}} &= [\langle a, b\rangle] \cdot_{\mathbb{Q}} [\langle 1, 1\rangle] \\ &= [\langle a \cdot 1, b \cdot 1\rangle] \\ &= [\langle a, b\rangle] \\ &= 1_{\mathbb{Q}} \end{aligned}$$

(b) Suppose that $q = [\langle a, b\rangle] \neq [\langle 0, 1\rangle]$. Then $a \neq 0$ and so $\langle b, a\rangle \in \mathbb{Z}\times\mathbb{Z}'$. Let $r = [\langle b, a\rangle]$. Then

$$\begin{aligned} q \cdot_{\mathbb{Q}} r &= [\langle a, b\rangle] \cdot_{\mathbb{Q}} [\langle b, a\rangle] \\ &= [\langle ab, ba\rangle] \\ &= [\langle 1, 1\rangle] \\ &= 1_{\mathbb{Q}}. \end{aligned}$$

To see that there is at most one such $r \in \mathbb{Q}$, suppose that also that $q \cdot_{\mathbb{Q}} r' = 1_{\mathbb{Q}}$. Then

$$\begin{aligned} r &= r \cdot_{\mathbb{Q}} 1_{\mathbb{Q}} \\ &= r \cdot_{\mathbb{Q}} (q \cdot_{\mathbb{Q}} r') \\ &= (r \cdot_{\mathbb{Q}} q) \cdot_{\mathbb{Q}} r' \\ &= (q \cdot_{\mathbb{Q}} r) \cdot_{\mathbb{Q}} r' \\ &= 1_{\mathbb{Q}} \cdot_{\mathbb{Q}} r' \\ &= r'. \end{aligned}$$

□

Definition 10.19. For any $0_{\mathbb{Q}} \neq q \in \mathbb{Q}$, q^{-1} is the unique element of $\mathbb{Q}$ such that $q \cdot_{\mathbb{Q}} q^{-1} = 1_{\mathbb{Q}}$.

Finally we want to define an order relation on $\mathbb{Q}$. [Note that if $b, d > 0$, then

$$\frac{a}{b} < \frac{c}{d} \quad \text{iff} \quad ad < cb.$$

Note that $[\langle a, b\rangle] = [\langle -a, -b\rangle]$, so each $q \in \mathbb{Q}$ can be represented as $[\langle a, b\rangle]$, where $b > 0$. This suggests that we make the following definition.]

Definition 10.20. Suppose that $r, s \in \mathbb{Q}$ and that $r = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$, where $b, d > 0$. Then

$$r \leq_{\mathbb{Q}} s \quad \text{iff} \quad ad < cb.$$

Lemma 10.21. *$<_{\mathbb{Q}}$ is well-defined.* □

Theorem 10.22. *$<_{\mathbb{Q}}$ is a linear order on $\mathbb{Q}$.* □

Definition 10.23. If $q \in \mathbb{Q}$, then

- q is *positive* iff $0_{\mathbb{Q}} <_{\mathbb{Q}} q$.
- q is *negative* iff $q <_{\mathbb{Q}} 0_{\mathbb{Q}}$.

Definition 10.24. If $q \in \mathbb{Q}$, then the *absolute value* of q is

$$\begin{aligned} |q| &= -q \quad \text{if q is negative} \\ &= q \quad \text{otherwise.} \end{aligned}$$

Remark 10.25. Clearly $\mathbb{Z}$ is not *literally* a subset of $\mathbb{Q}$. However, $\mathbb{Q}$ does contain an "isomorphis copy" of $\mathbb{Z}$.

Definition 10.26. Let $E \colon \mathbb{Z} \to \mathbb{Q}$ be the function defined by

$$E(a) = [\langle a, 1\rangle].$$

Theorem 10.27. *E is an injection of $\mathbb{Z}$ into $\mathbb{Q}$ which satisfies the following conditions for all $a, b \in \mathbb{Z}$*

- $E(a+b) = E(a) +_{\mathbb{Q}} E(b)$
- $E(ab) = E(a) \cdot_{\mathbb{Q}} E(b)$
- $E(0) = 0_{\mathbb{Q}}$ *and* $E(1) = 1_{\mathbb{Q}}$.
- $a < b$ *iff* $E(a) <_{\mathbb{Q}} E(b)$. $\square$

Notation From now on, we write $+, \cdot, <, 0, 1$ instead of $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, <_{\mathbb{Q}}, 0_{\mathbb{Q}}, 1_{\mathbb{Q}}$.