8 The Ordering on $\omega$

Definition 8.1. $\in_\omega$ is the binary relation on $\omega$ defined by

$$\in_\omega = \{ (m, n) \in \omega \times \omega \mid m < n \}.$$

In this section, we shall prove:

Theorem 8.2. $\in_\omega$ is a linear order on $\omega$.

Definition 8.3. A set $A$ is transitive iff whenever $x \in a \in A$, then $x \in A$.

Example 8.4.

1. $\{\{\emptyset\}\}$ is not transitive, since $\emptyset \in \{\emptyset\} \in \{\{\emptyset\}\}$ but $\emptyset \notin \{\{\emptyset\}\}$.

2. $\{\emptyset, \{\emptyset\}\}$ is transitive.

Lemma 8.5. If $n \in \omega$, then $n$ is transitive.

Proof. It is enough to prove that the set

$$T = \{ n \in \omega \mid n \text{ is transitive} \}$$

is inductive. First $\emptyset$ is trivially transitive and so $\emptyset \in T$. Next suppose that $n \in T$ and that

$$x \in a \in n^+ = n \cup \{n\}.$$

There are two cases to consider.

Case 1 Suppose that $a \in n$. Since $n \in T$ and $x \in a \in n$, it follows that $x \in n$ and so $x \in n^+$.

Case 2 Suppose that $a = n$. Then $x \in n$ and so $x \in n^+$. Thus in both cases, $x \in n^+$. Hence $n^+ \in T$. □

Remark 8.6. In other words, if $a, b, c \in \omega$, then

$$a \in b \quad \text{and} \quad b \in c \quad \text{implies} \quad a \in c.$$

Thus $\in_\omega$ is a transitive relation on $\omega$.

Lemma 8.7.

(a) For any $n, m \in \omega$, $m \in n$ iff $m^+ \in n^+$.

(b) For all $n \in \omega$, $n \notin n$. 
Proof. (a) First suppose that \( m^+ \in n^+ \). Then
\[
m \in m^+ \in n^+ = n \cup \{n\}.
\]
There are two cases to consider.

**Case 1** Suppose that \( m^+ \in n \). Then \( m \in m^+ \in n \) and so \( m \in n \).

**Case 2** Suppose that \( m^+ = n \). Then \( m \in n \).

Thus in either case, \( m \in n \).

To prove the converse, we use induction. In other words, we prove that
\[
T = \{ n \in \omega \mid (\forall m \in n) \ m^+ \in n^+ \}
\]
is inductive. First \( \emptyset \in T \) vacuously. Next suppose that \( n \in T \). We must prove that
\[
(\ast) \text{ if } m \in n^+, \text{ then } m^+ \in n^{++}.
\]
So suppose that \( m \in n^+ = n \cup \{n\} \). If \( m = n \) then
\[
m^+ = n^+ \in n^{++} = n^+ \cup \{n^+\}.
\]
Otherwise, \( m \in n \) and so since \( n \in T \),\( m \in n^+ \in n^{++} \)
and so \( m^+ \in n^{++} \). Hence \( n^+ \in T \).

(b) It is enough to show that \( S = \{ n \in \omega \mid n \notin n \} \)
is inductive. Clearly \( \emptyset \in S \). Next suppose that \( n \in S \). For the sake of contradiction, assume that \( n^+ \in n^+ \). By (a), \( n \in n \), which contradicts the fact that \( n \in S \). Thus \( n^+ \notin n^+ \) and so \( n^+ \in S \).

**Lemma 8.8.** For any \( n, m \in \omega \) at most one of the following holds:
\[
m, m = n, \ n \in m.
\]

*Proof.* By Lemma 8.7 (b), if two hold, then we must have that \( m \in n \) and \( n \in m \). By Lemma 8.5, \( m \in m \), which contradicts Lemma 8.7 (b). \( \square \)

**Lemma 8.9.** For any \( n, m \in \omega \) at least one of the following holds:
\[
m, m = n, \ n \in m.
\]

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Proof. It is enough to show that
\[ T = \{ n \in \omega \mid (\forall m \in \omega) (m \in n \text{ or } m = n \text{ or } n \in m) \} \]
is inductive.

**Exercise 8.10.** Prove that for all \( m \in \omega \), \( m = \emptyset \) or \( \emptyset \in m \).

**Hint** Argue by induction on \( m \).

Thus \( \emptyset \in T \). Next suppose that \( n \in T \). Let \( m \in \omega \) be arbitrary. Since \( n \in T \), we have that
\[
m \in n \quad \text{or} \quad m = n \quad \text{or} \quad n \in m.
\]
If \( m \in n \) or \( m = n \), then \( m \in n^+ = n \cup \{n\} \). If \( n \in m \), then Lemma 8.7 (a) implies
\[
n^+ \in m^+ = m \cup \{m\}
\]
and so \( n^+ \in m \) or \( n^+ = m \). In either case, we have that
\[
m \in n^+ \quad \text{or} \quad m = n^+ \quad \text{or} \quad n^+ \in m.
\]
Thus \( n^+ \in T \).

This completes the proof that \( \in_\omega \) is a linear order on \( \omega \).

**Notation** (Different from Enderton) From now on, if \( m, n \in \omega \), then we use the following notation interchangeably:
\[
 m \in n \quad \text{iff} \quad m < n
\]
\[
 m \in \overrightarrow{n} \quad \text{iff} \quad m \leq n
\]

**Exercise 8.11.** Let \( < \) be a linear order on a \( A \). If \( a, b \in A \) satisfy \( a \leq b \) and \( b \leq a \) then \( a = b \).

**Theorem 8.12 (Well-ordering of \( \omega \)).** If \( \emptyset \neq A \subseteq \omega \), then there exists \( m \in A \) such that \( m \leq a \) for all \( a \in A \); ie \( m \in a \) or \( m = a \) for all \( a \in A \).

**Proof.** Assume that no such element exists. Define
\[
 B = \{ n \in \omega \mid (\forall k \in n) k \in \omega \setminus A \}
\]
We shall prove that \( B \) is inductive. Clearly \( \emptyset \in B \) vacuously. Next suppose that \( n \in B \). Thus (i) If \( k \in n \), then \( k \notin A \). Suppose that \( n^+ \notin B \). Then there exists \( k \in n^+ = n \cup \{n\} \) such that \( k \in A \). By (i) we must have that (ii) \( n \in A \) Now let \( a \in A \) be arbitrary. By Trichotomy, either
\[
a \in n \quad \text{or} \quad a = n \quad \text{or} \quad n \in a.
\]
By (i), \( a \notin n \). Thus for all \( a \in A \), \( a = n \) or \( n \in a \), contradicting our assumption. Hence \( n^+ \in B \). By Induction, \( B = \omega \). But this means that \( A = \emptyset \), which is a contradiction. \( \square \)
Theorem 8.13 (Strong Induction Principle for $\omega$). Let $A \subseteq \omega$ and suppose that for every $n \in \omega$,
\[
(*) \text{ if } m \in A \text{ for all } m < n, \text{ then } n \in A.
\]
Then $A = \omega$.

Proof. Suppose that $A \neq \omega$. Then $\omega \smallsetminus A \neq \emptyset$ and so there exists a least element $k \in \omega \smallsetminus A$. Since $k$ is the least such element, it follows that $m \in A$ for all $m < k$. Then $(*)$ implies that $k \in A$, which is a contradiction. \qed

Definition 8.14. Suppose that $<_A,<_B$ are linear orders on $A,B$ respectively. Then a function $f : A \to B$ is order-preserving iff for all $a_1,a_2 \in A$,
\[
(*) \text{ if } a_1 <_A a_2, \text{ then } f(a_1) <_B f(a_2).
\]

Exercise 8.15. Suppose that $f : A \to B$ is order-preserving. Then the following statements are true.

- If $a_1,a_2 \in A$, then $a_1 <_A a_2$ iff $f(a_1) <_B f(a_2)$.
- $f$ is an injection.
- If $f$ is a bijection, then $f^{-1} : B \to A$ is also order-preserving.

Theorem 8.16. If $f : \omega \to \omega$ is order-preserving, then $f(n) \geq n$ for all $n \in \omega$.

Proof. If not, then
\[
C = \{n \in \omega \mid f(n) < n\} \neq \emptyset.
\]
Let $k \in C$ be the least element. Then $f(k) < k$. Since $f$ is order-preserving, this implies that $f(f(k)) < f(k)$. Hence $f(k) \in C$, which contradicts the minimality of $k$. \qed

Corollary 8.17. If $f : \omega \to \omega$ is an order-preserving bijection, then $f(n) = n$ for all $n \in \omega$.

Proof. Since $f$ is order-preserving, $f(n) \geq n$ for all $n \in \omega$. Since $f^{-1}$ is order-preserving, $f^{-1}(n) \geq n$ for all $n \in \omega$. This implies that $f(f^{-1}(n)) \geq f(n)$ and so $n \geq f(n)$ for all $n \in \omega$. Hence $f(n) = n$ for all $n \in \omega$. \qed

Remark 8.18. The above remark fails for $\mathbb{Z}$. For example, the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z + 1$ is an order-preserving bijection.

Exercise 8.19. Suppose that $A$ is a transitive set. Then

- $\mathcal{P}(A)$ is also a transitive set.
- $A \subseteq \mathcal{P}(A)$. 

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Informal discussion of $V = \bigcup_{\alpha \in \text{On}} V_\alpha$, where

\[
V_0 = \emptyset \\
V_{\alpha+1} = \mathcal{P}(V_\alpha) \\
V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \text{ for lim } \lambda
\]

Axioms so far...  
- **Extensionality**  
  By the transitivity of $V_\alpha$  
- **Empty Set**  
  $\emptyset \in V_1$  
- **Subset Axiom**  
  If $A \in V_\alpha$ and $B \subseteq A$, then $B \in V_{\alpha+1}$  
- **Union Axiom**  
  If $A \in V_\alpha$, then $\bigcup A \in V_{\alpha+1}$  
- **Pairing Axiom**  
  If $A, B \in V_\alpha$, then $\{A, B\} \in V_{\alpha+1}$  
- **Powerset Axiom**  
  If $A \in V_\alpha$, then $\mathcal{P}(A) \in V_{\alpha+2}$  
- **Infinity**  
  $\omega \in V_{\omega+1}$

The following results will be crucial in our construction of $\mathbb{Z}$.

**Theorem 8.20.** For any $m, n, p \in \omega$, we have that

\[m < n \iff m + p < n + p.\]

*Proof.* Reading Exercise, Enderton p. 85-86.

**Corollary 8.21 (Cancellation Law).** For any $m, n, p \in \omega$, if $m + p = n + p$, then $m = n$.

*Proof.* Suppose that $m + p = n + p$. By Trichotomy, if $m \neq n$, then either $m < n$ or $n < m$. By Theorem 8.20, if $m < n$ then $m + p < n + p$, which contradicts Trichotomy. Similarly, if $n < m$, then $n + p < m + p$, which also contradicts Trichotomy. Hence $m = n$.

The following results will be crucial in our construction of $\mathbb{Q}$.

**Theorem 8.22.** If $m, n, p \in \omega$ and $p \neq 0$, then

\[m < n \iff m \cdot p < n \cdot p.\]

*Proof.* Reading Exercise, Enderton p. 85-86.

**Corollary 8.23 (Cancellation Law).** If $m, n, p \in \omega$ and $p \neq 0$, then $m \cdot p = n \cdot p$ implies $m = n$. 

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