8 The Ordering on ω

Definition 8.1. $\in_\omega$ is the binary relation on ω defined by

$$\in_\omega = \{\langle m, n\rangle \in \omega \times \omega \mid m \in n\}.$$

In this section, we shall prove:

Theorem 8.2. *$\in_\omega$ is a linear order on ω.*

Definition 8.3. A set A is *transitive* iff whenever $x \in a \in A$, then $x \in A$.

Example 8.4.

1. $\{\{\emptyset\}\}$ is *not* transitive, since $\emptyset \in \{\emptyset\} \in \{\{\emptyset\}\}$ but $\emptyset \notin \{\{\emptyset\}\}$.
2. $\{\emptyset, \{\emptyset\}\}$ is transitive.

Lemma 8.5. *If $n \in \omega$, then n is transitive.*

Proof. It is enough to prove that the set

$$T = \{n \in \omega \mid n \text{ is transitive}\}$$

is inductive. First $\emptyset$ is trivially transitive and so $\emptyset \in T$. Next suppose that $n \in T$ and that

$$x \in a \in n^+ = n \cup \{n\}.$$

There are two cases to consider.

Case 1 Suppose that $a \in n$. Since $n \in T$ and $x \in a \in n$, is follows that $x \in n$ and so $x \in n^+$.

Case 2 Suppose that $a = n$. Then $x \in n$ and so $x \in n^+$.

Thus in both cases, $x \in n^+$. Hence $n^+ \in T$. □

Remark 8.6. In other words, if $a, b, c \in \omega$, then

$$a \in b \quad \text{and} \quad b \in c \quad \text{implies} \quad a \in c.$$

Thus $\in_\omega$ is a transitive relation on ω.

Lemma 8.7.

(a) For any $n, m \in \omega$,

$$m \in n \quad \textit{iff} \quad m^+ \in n^+.$$

(b) For all $n \in \omega$, $n \notin n$.

Proof. (a) First suppose that $m^+ \in n^+$. Then

$$m \in m^+ \in n^+ = n \cup \{n\}.$$

There are two cases to consider.

Case 1 Suppose that $m^+ \in n$. Then $m \in m^+ \in n$ and so $m \in n$.

Case 2 Suppose that $m^+ = n$. Then $m \in n$.

Thus in either case, $m \in n$.

To prove the converse, we use induction. In other words, we prove that

$$T = \{n \in \omega \mid (\forall m \in n)\ m^+ \in n^+\}$$

is inductive. First $\emptyset \in T$ vacuously. Next suppose that $n \in T$. We must prove that

$$(*) \quad \text{if } m \in n^+, \text{ then } m^+ \in n^{++}.$$

So suppose that $m \in n^+ = n \cup \{n\}$. If $m = n$ then

$$m^+ = n^+ \in n^{++} = n^+ \cup \{n^+\}.$$

Otherwise, $m \in n$ and so since $n \in T$m

$$m^+ \in n^+ \subseteq n^{++}$$

and so $m^+ \in n^{++}$. Hence $n^+ \in T$.

(b) It is enough to show that

$$S = \{n \in \omega \mid n \notin n\}$$

is inductive. Clearly $\emptyset \in S$. Next suppose that $n \in S$. FOr the sake of contradiction, assume that $n^+ \in n^+$. By (a), $n \in n$, which contradicts the fact that $n \in S$. Thus $n^+ \notin n^+$ and so $n^+ \in S$. □

Lemma 8.8. *For any $n, m \in \omega$* at most *one of the following holds:*

$$m \in n, \quad m = n, \quad n \in m.$$

Proof. By Lemma 8.7 (b), if two hold, then we must have that $m \in n$ and $n \in m$. By Lemma 8.5, $m \in m$, which contradicts Lemma 8.7 (b). □

Lemma 8.9. *For any $n, m \in \omega$ at least one of the following holds:*

$$m \in n, \quad m = n, \quad n \in m.$$

Proof. It is enough to show that

$$T = \{n \in \omega \mid (\forall m \in \omega)\ (m \in n \text{ or } m = n \text{ or } n \in m)\}$$

is inductive.

Exercise 8.10. Prove that for all $m \in \omega$, $m = \emptyset$ or $\emptyset \in m$.

Hint Argue by induction on m.

Thus $\emptyset \in T$. Next suppose that $n \in T$. Let $m \in \omega$ be arbitrary. Since $n \in T$, we have that

$$m \in n \quad \text{or} \quad m = n \quad \text{or} \quad n \in m.$$

If $m \in n$ or $m = n$, then $m \in n^+ = n \cup \{n\}$. If $n \in m$, then Lemma 8.7 (a) implies

$$n^+ \in m^+ = m \cup \{m\}$$

and so $n^+ \in m$ or $n^+ = m$. In either case, we have that

$$m \in n^+ \quad \text{or} \quad m = n^+ \quad \text{or} \quad n^+ \in m.$$

Thus $n^+ \in T$. □

This completes the proof that $\in_\omega$ is a linear order on ω.

Notation (Different from Enderton) From now on, if $m, n \in \omega$, then we use the following notation interchangably:

$$m \in n \quad \text{iff} \quad m < n$$
$$m \overset{\in}{-} n \quad \text{iff} \quad m \leq n$$

Exercise 8.11. Let $<$ be a linear order on a A. If $a, b \in A$ satisfy $a \leq b$ and $b \leq a$ then $a = b$.

Theorem 8.12 (Well-ordering of ω). *If $\emptyset \neq A \subseteq \omega$, then there exists $m \in A$ such that $m \leq a$ for all $a \in A$; ie $m \in a$ or $m = a$ for all $a \in A$.*

Proof. Assume that no such element exists. Define

$$B = \{n \in \omega \mid (\forall k \in n)\ k \in \omega \smallsetminus A\}$$

We shall prove that B is inductive. Clearly $\emptyset \in B$ vacuously. Next suppose that $n \in B$. Thus (i) If $k \in n$, then $k \notin A$. Suppose that $n^+ \notin B$. Then there exists $k \in n^+ = n \cup \{n\}$ such that $k \in A$. By (i) we must have that (ii) $n \in A$ Now let $a \in A$ be arbitrary. By Trichotomy, either

$$a \in n \quad \text{or} \quad a = n \quad \text{or} \quad n \in a.$$

By (i), $a \notin n$. Thus for all $a \in A$, $a = n$ or $n \in a$, contradicting our assumption. Hence $n^+ \in B$. By Induction, $B = \omega$. But this means that $A = \emptyset$, which is a contradiction. □

Theorem 8.13 (Strong Induction Principle for ω). *Let $A \subseteq \omega$ and suppose that for every $n \in \omega$,*

$$(*) \text{ if } m \in A \text{ for all } m < n, \text{ then } n \in A.$$

Then $A = \omega$.

Proof. Suppose that $A \neq \omega$. Then $\omega \smallsetminus A \neq \emptyset$ and so there exists a least element $k \in \omega \smallsetminus A$. Since k is the least such element, it follows that $m \in A$ for all $m < k$. Then (*) implies that $k \in A$, which is a contradiction. □

Definition 8.14. Suppose that $<_A, <_B$ are linear orders on A, B respectively. Then a function $f\colon A \to B$ is *order-preserving* iff for all $a_1, a_2 \in A$,

$$(*) \text{ if } a_1 <_A a_2, \text{ then } f(a_1) <_B f(a_2).$$

Exercise 8.15. Suppose that $f\colon A \to B$ is order-preserving. Then the following statements are true.

- If $a_1, a_2 \in A$, then $a_1 <_A a_2$ iff $f(a_1) <_B f(a_2)$.
- f is an injection.
- If f is a bijection, then $f^{-1}\colon B \to A$ is also order-preserving.

Theorem 8.16. *If $f\colon \omega \to \omega$ is order-preserving, then $f(n) \geq n$ for all $n \in \omega$.*

Proof. If not, then

$$C = \{n \in \omega \mid f(n) < n\} \neq \emptyset.$$

Let $k \in C$ be the least element. Then $f(k) < k$. Since f is order-preserving, this implies that $f(f(k)) < f(k)$. Hence $f(k) \in C$, which contradicts the minimality of k. □

Corollary 8.17. *If $f\colon \omega \to \omega$ is an order-preserving bijection, then $f(n) = n$ for all $n \in \omega$.*

Proof. Since f is order-preserving, $f(n) \geq n$ for all $n \in \omega$. Since f^{-1} is order-preserving, $f^{-1}(n) \geq n$ for all $n \in \omega$. This implies that $f(f^{-1}(n)) \geq f(n)$ and so $n \geq f(n)$ for all $n \in \omega$. Hence $f(n) = n$ for all $n \in \omega$. □

Remark 8.18. The above remark fails for $\mathbb{Z}$. For example, the function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z + 1$ is an order-preserving bijection.

Exercise 8.19. Suppose that A is a transitive set. Then

- $\mathcal{P}(A)$ is also a transitive set.
- $A \subseteq \mathcal{P}(A)$.

Informal discussion of $V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$, where

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \quad \text{for lim } \lambda \end{aligned}$$

Axioms so far...

Extensionality	By the transitivity of V_α
Empty Set	$\emptyset \in V_1$
Subset Axiom	If $A \in V_\alpha$ and $B \subseteq A$, then $B \in V_{\alpha+1}$
Union Axiom	If $A \in V_\alpha$, then $\bigcup A \in V_{\alpha+1}$
Pairing Axiom	If $A, B \in V_\alpha$, then $\{A, B\} \in V_{\alpha+1}$
Powerset Axiom	If $A \in V_\alpha$, then $\mathcal{P}(A) \in V_{\alpha+2}$
Infinity	$\omega \in V_{\omega+1}$

The following results will be crucial in our construction of $\mathbb{Z}$.

Theorem 8.20. *For any $m, n, p \in \omega$, we have that*

$$m < n \quad \textit{iff} \quad m + p < n + p.$$

Proof. Reading Exercise, Enderton p. 85-86. ☐

Corollary 8.21 (Cancellation Law). *For any $m, n, p \in \omega$, if $m + p = n + p$, then $m = n$.*

Proof. Suppose that $m + p = n + p$. By Trichotomy, if $m \neq n$, then either $m < n$ or $n < m$. By Theorem 8.20, if $m < n$ then $m + p < n + p$, which contradict Trichotomy. Similarly, if $n < m$, then $n + p < m + p$, which also contradicts Trichotomy. Hence $m = n$. ☐

The following resuls will be crucial in our construction of $\mathbb{Q}$.

Theorem 8.22. *If $m, n, p \in \omega$ and $p \neq 0$, then*

$$m < n \quad \textit{iff} \quad m \cdot p < n \cdot p.$$

Proof. Reading Exercise, Enderton p. 85-86. ☐

Corollary 8.23 (Cancellation Law). *If $m, n, p \in \omega$ and $p \neq 0$, then $m \cdot p = n \cdot p$ implies $m = n$.* ☐