4 Relations

$$\text{Functions} \equiv \text{Operations}$$

$$\text{Relations} \equiv \text{Properties}$$

Remark 4.1. In this course, we will only consider *binary* relations.

Example 4.2. Consider the order relation on $A = \{0, 1, 2\}$ of *less than*. Then $0 < 1$, $0 < 2$, and $1 < 2$. But $2 \not< 1$, *etc.* We shall define the order relation to be

$$<= \{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 2\rangle\}.$$

Example 4.3. Consider the relation E on $A = \{0, 1, 2\}$ defined by

$$xEy \text{ iff } x - y \text{ is even.}$$

Then we shall define

$$E = \{\langle 0, 2\rangle, \langle 2, 0\rangle, \langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\}.$$

Definition 4.4. Let A be a set. Then R is a *relation* on A iff $R \subseteq A \times A$.

Remark 4.5. In this case, we would usually write xRy instead of $\langle x, y\rangle \in R$. *eg* "$0 < 2$" is more natural than "$\langle 0, 2\rangle \in <$".

Definition 4.6 (More General Definition). A *relation* is a set of ordered pairs.

eg $R = \{\langle 0, 3\rangle, \langle 1, 4\rangle, \langle 2, 5\rangle, \langle 3, 6\rangle\}$ is a relation.

Definition 4.7. Let R be a relation. Then we define $\operatorname{dom} R$, $\operatorname{ran} R$, and $\operatorname{fld} R$ by

$$x \in \operatorname{dom} R \text{ iff there exists } y \text{ such that } \langle x, y\rangle \in R$$

$$x \in \operatorname{ran} R \text{ iff there exists } y \text{ such that } \langle y, x\rangle \in R$$

$$\operatorname{fld} R = \operatorname{dom} R \cup \operatorname{ran} R$$

Remark 4.8. Thus R is a relation on $\operatorname{fld} R$.

Example 4.9. Let $R = \{\langle l, n\rangle \mid l \text{ is the } n^{\text{th}} \text{ letter of the alphabet}\}$. Then $\operatorname{dom} R = \{l \mid l \text{ is a letter of the alphabet }\}$ and $\operatorname{ran} R = \{1, 2, \ldots, 26\}$.

5 Equivalence Relations

Definition 5.1. Let R be a binary relation on A.

- R is *reflexive* iff xRx for all $x \in A$.
- R is *symmetric* iff for all $x, y \in A$, if xRy then yRx.
- R is *transitive* iff for all $x, y, z \in A$, if xRy and yRz then xRz.
- R is an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Example 5.2. Let E be the relation on $\mathbb{Z}$ defined by

$$xEy \quad \text{iff} \quad 3 | x - y$$

Then E is an equivalence relation.

Proof. We check that E is reflexive, symmetric, and transitive.

- If $x \in \mathbb{Z}$, then $3|x - x = 0$ since $0 = 3 \cdot 0$. Thus xEx.
- Suppose that xEy. Then $3|x - y$ and so there exists $z \in \mathbb{Z}$ such that $x - y = 3 \cdot z$. Then $y - x = 3 \cdot (-z)$ and so $3|y - x$. Thus yEx.
- Suppose that xEy and yEz. Thus $3|x - y$ and $3|y - z$. Hence there exist $a, b \in \mathbb{Z}$ such that $x - y = 3a$ and $y - z = 3b$. It follows that

$$\begin{aligned} x - z &= (x - y) + (y - z) \\ &= 3a + 3b \\ &= 3(a + b) \end{aligned}$$

and so $3|x - z$. Thus xEz. □

Exercise 5.3. Define a relation R on $\mathbb{N}\times\mathbb{N}$ by

$$\langle a, b\rangle R\langle c, d\rangle \quad \text{iff} \quad a + d = c + b$$

Prove that R is an equivalence relation on $\mathbb{N}\times\mathbb{N}$.

Definition 5.4. Suppose that R is an equivalence relation on A. Then, for each $x \in A$, the R–equivalence class of x is

$$[x]_R = \{y \in A \mid xRy\}.$$

Theorem 5.5. *Suppose that R is an equivalence relation on A.*

- $x \in [x]_R$ *for each* $x \in A$.

- *If $x, y \in A$ and $[x]_R \cap [y]_R \neq \emptyset$, then $[x]_R = [y]_R$.*

Proof.

- If $x \in A$, then xRx and so $x \in [x]_R$.
- Suppose that $z \in [x]_R \cap [y]_R$; *ie* xRz and yRz. Since yRz, it follows that zRy. Since xRz and zRy, it follows that xRy. We claim that $[y]_R \subseteq [x]_R$. To see this, suppose that $t \in [y]_R$; *ie* yRt. Since xRy and yRt, it follows that xRt and so $t \in [x]_R$. Similarly, $[x]_R \subseteq [y]_R$ and so $[x]_R = [y]_R$. □

Definition 5.6. Let A be a set. Then Π is a *partition* of A iff:

- Π is a set of nonempty subsets of A.
- If $B, C \in \Pi$ are distinct, then $B \cap C = \emptyset$.
- $\bigcup \Pi = A$.

Example 5.7. $\Pi = \{0, 2\}, \{1, 3\}, \{4\}\}$ is a partition of $A = \{0, 1, 2, 3, 4\}$.

Theorem 5.8. *Suppose that R is an equivalence relation on A. Then*

$$\{[x]_R \mid x \in A\}$$

is a partition of A. □

Example 5.9. Let E be the equivalence relation on $\mathbb{Z}$ defined by

$$xEy \quad \text{iff} \quad 3|x - y$$

Then

$$[0]_E = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$[1]_E = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$[2]_E = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$$

Thus $\{[0]_E, [1]_E, [2]_E\}$ is the corresponding partition of $\mathbb{Z}$.

Theorem 5.10. *Suppose that Π is a partition of the set A. Define the relation R_Π by:*

$$xR_\Pi y \text{ iff there exists } B \in \Pi \text{ such that } x, y \in B.$$

Then:

- *R_Π is an equivalence relation on A.*
- *Π is the set of R_Π–equivalence classes.*

Proof. Exercise! □

Definition 5.11. Suppose that R is an equivalence relation on A. Then the *quotient set* is

$$A/R = \{[x]_R \mid x \in A\}$$

and the *natural map* / *canonical map* $\varphi\colon A \to A/R$ is defined by

$$\varphi(x) = [x]_R.$$