11 The compactness theorem

Question 11.1. Suppose that $\Sigma$ is an infinite set of wffs and that $\Sigma \models \tau$. Does there necessarily exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$?

A positive answer follows from the following result...

Theorem 11.2 (The Compactness Theorem). Let $\Sigma$ be a set of wffs. If every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable, then $\Sigma$ is satisfiable.

Definition 11.3. A set $\Sigma$ of wffs is finitely satisfiable iff every finite subset $\Sigma_0 \subset \Sigma$ is satisfiable.

Theorem 11.4 (The Compactness Theorem). If $\Sigma$ is a finitely satisfiable set of wffs, then $\Sigma$ is satisfiable.

Before proving the compactness theorem, we present a number of its applications.

Corollary 11.5. If $\Sigma \models \tau$, then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$.

Proof. Suppose not. Then for every finite subset $\Sigma_0 \subseteq \Sigma$, we have that $\Sigma_0 \not\models \tau$ and hence $\Sigma_0 \cup \{\neg \tau\}$ is satisfiable. Thus $\Sigma \cup \{\neg \tau\}$ is finitely satisfiable. By the Compactness Theorem, $\Sigma \cup \{\neg \tau\}$ is satisfiable. But this means that $\Sigma \not\models \tau$, which is a contradiction.\qed

12 A graph-theoretic application

Definition 12.1. Let $E$ be a binary relation on the set $V$. Then $\Gamma = \langle V, E \rangle$ is a graph iff:

1. $E$ is irreflexive; and
2. $E$ is symmetric.

Example 12.2. Let $V = \{0, 1, 2, 3, 4\}$ and let $E = \{(i, j) \mid j = i + 1 \mod 5\}$. This is called the cycle of length five.

Definition 12.3. Let $k \geq 1$. Then the graph $\Gamma = \langle V, E \rangle$ is $k$-colorable iff there exists a function $\chi: V \to \{1, 2, \ldots, k\}$ such that for all $a, b \in V$,

(*) if $aEb$, then $\chi(a) \neq \chi(b)$.

Example 12.4. Any cycle of even length is two-colorable. Any cycle of odd length is three-colorable but not two-colorable.

Theorem 12.5 (Erdős). A countable graph $\Gamma = \langle V, E \rangle$ is $k$-colorable iff every finite subgraph $\gamma_0 \subset \Gamma$ is $k$-colorable.
\textbf{Proof.} Suppose that $\Gamma$ is $k$-colorable and let $\chi: V \to \{1, 2, \ldots, k\}$ is any $k$-coloring. Let $\Gamma_0 = \langle V_0, E_0 \rangle$ be any finite subgraph of $\Gamma$. Then $\chi_0 = \chi|V_0$ is a $k$-coloring of $\Gamma_0$.

$\leftarrow$ In this direction we use the Compactness Theorem.

\textbf{Step 1} We choose a suitable propositional language. The idea is to have a sentence symbol for every decision we must make. So our language has sentence symbols:

$C_{v,i}$ for each $v \in V$, $1 \leq i \leq k$.

(The intended meaning of $C_{v,i}$ is: “color vertex $v$ with color $i$.”)

\textbf{Step 2} We write down a suitable set of wffs which imposes a suitable set of constraints on our truth assignments. Let $\Sigma$ be the set of wffs of the following forms:

\begin{enumerate}
  \item $C_{v,1} \lor C_{v,2} \lor \ldots \lor C_{v,k}$ for each $v \in V$.
  \item $\neg(C_{v,i} \land C_{v,j})$ for each $v \in V$ and $1 \leq i \neq j \leq k$.
  \item $\neg(C_{v,i} \land C_{w,i})$ for each pair $v, w \in V$ of adjacent vertices and each $1 \leq i \leq k$.
\end{enumerate}

\textbf{Step 3} We check that we have chosen a suitable set of wffs.

\textbf{Claim 12.6.} Suppose that $\varphi$ is a truth assignment which satisfies $\Sigma$. Then we can define a $k$-coloring $\chi: \Gamma \to \{1, \ldots, k\}$ by

$\chi(v) = i$ if $v(C_{v,i}) = T$.

\textbf{Proof.} By (a) and by (b), for each $v \in V$, there exists a unique $1 \leq i \leq k$ such that $\varphi(C_{v,i}) = T$. Thus $\chi: V \to \{1, \ldots\}$ is a function. By (c), if $v, w \in V$ are adjacent, then $\chi(v) \neq \chi(w)$. Hence $\chi$ is a $k$-coloring. \hfill $\Box$

\textbf{Step 4} We next prove that $\Sigma$ is finitely satisfiable. So let $\Sigma_0 \subseteq \Sigma$ be any finite subset. Let $V_0 \subseteq V$ be the finite set of vertices that are mentioned in $\Sigma_0$. Then the finite subgraph $\Gamma_0 = \langle V_0, E_0 \rangle$ is $k$-colorable. Let

$\chi: V_0 \to \{1, \ldots, k\}$

be a $k$-coloring of $\Gamma_0$. Let $\varphi_0$ be a truth assignment such that if $v \in V_0$ and $1 \leq i \leq k$, then

$\varphi_0(C_{v,i}) = T \text{ iff } \chi_0(v) = i.$

Clearly $\varphi_0$ satisfies $\Sigma_0$.

By the Compactness Theorem, $\Sigma$ is satisfiable. Hence $\Gamma$ is $k$-colorable. \hfill $\Box$
13 Extending partial orders

**Theorem 13.1.** Let \((A, \prec)\) be a countable partial order. Then there exists a linear ordering \(<\) of \(A\) which extends \(\prec\).

**Proof.** We work with the propositional language which has sentence symbols

\[ L_{a,b} \quad \text{for } a \neq b \in A \]

Let \(\Sigma\) be the following set of wffs:

(a) \(L_{a,b} \lor L_{b,a}\) for \(a \neq b \in A\)
(b) \(\neg(L_{a,b} \land L_{b,a})\) for \(a \neq b \in A\)
(c) \(((L_{a,b} \land L_{b,c}) \rightarrow L_{a,c})\) for distinct \(a, b, c \in A\)
(d) \(L_{a,b}\) for distinct \(a, b \in A\) with \(a \prec b\).

**Claim 13.2.** Suppose that \(v\) is a truth assignment which satisfies \(\Sigma\). Define the binary relation \(<\) on \(A\) by

\[ a < b \iff v(L_{a,b}) = T. \]

Then \(<\) is a linear ordering of \(A\) which extends \(\prec\).

**Proof.** Clearly \(<\) is irreflexive. By (a) and (b), \(<\) has the trichotomy property. By (c), \(<\) is transitive. Finally, by (d), \(<\) extends \(\prec\). \(\square\)

Next we prove that \(\Sigma\) is finitely satisfiable. So let \(\Sigma_0 \subseteq \Sigma\) be any finite subset. Let \(A_0 \subseteq A\) be the finite set of elements that are mentioned in \(\Sigma_0\) and consider the partial order \((A_0, \prec_0)\). Then there exists a partial ordering \(\prec_0\) of \(A_0\) extending \(\prec_0\). Let \(v_0\) be the truth assignment such that if \(a \neq b \in A_0\), then

\[ v_0(L_{a,b}) = T \iff a \leq_0 b. \]

Clearly \(v_0\) satisfies \(\Sigma_0\).

By the compactness theorem, \(\Sigma\) is satisfiable. Hence there exists a linear ordering \(<\) of \(A\) which extends \(\prec\). \(\square\)

14 Hall’s Theorem

**Definition 14.1.** Suppose that \(S\) is a set and that \((S_i \mid i \in I)\) is an indexed collection of (not necessarily distinct) subsets of \(S\). A system of distinct representatives is a choice of elements \(x_i \in S_i\) for \(i \in I\) such that if \(i \neq j \in I\), then \(x_i \neq x_j\).
Example 14.2. Let $S = \mathbb{N}$ and let $\langle S_n \mid n \in \mathbb{N} \rangle$ be defined by

$$S_n = \{n, n + 1\}$$

Thus $S_0 = \{0, 1\}$, $S_1 = \{1, 2\}$, $\ldots$ Then we can take $x_i = i \in S_i$.

Theorem 14.3 (Hall’s Matching Theorem (1935)). Let $S$ be any set and let $n \in \mathbb{N}^+$. Let $\langle S_1, S_2, \ldots, S_n \rangle$ be an indexed collection of subsets of $S$. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:

$(H)$ For every $1 \leq k \leq n$ and choice of $k$ distinct indices $1 \leq i_1, \ldots, i_k \leq n$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Challenge: Prove this!

Problem 14.4. State and prove an infinite analogue of Hall’s Matching Theorem.

First Attempt Let $S$ be any set and let $\langle S_n \mid n \in \mathbb{N}^+ \rangle$ be an indexed collection of subsets of $S$. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:

$(H^*)$ For every $k \in \mathbb{N}^+$ and choice of $k$ distinct indices $i_1, \ldots, i_k \in \mathbb{N}$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Counterexample Take $S_1 = \mathbb{N}$, $S_2 = \{0\}$, $S_3 = \{1\}$, $\ldots$, $S_n = \{n - 2\}$, $\ldots$ Clearly $(H^*)$ is satisfied and yet there is no system of distinct representatives.

Question 14.5. Where does the compactness argument break down?

Theorem 14.6 (Infinite Hall’s Theorem). Let $S$ be any set and let $\langle S_n \mid n \in \mathbb{N}^+ \rangle$ be an indexed collection of finite subsets of $S$. Then a necessary and sufficient condition for the existence of a system of distinct representatives is:

$(H^*)$ For every $k \in \mathbb{N}^+$ and choice of $k$ distinct indices $i_1, \ldots, i_k \in \mathbb{N}$, we have $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$.

Proof. We work with the propositional language with sentence symbols

$$C_{n,x}. \ n \in \mathbb{N}^+, \ x \in S_n.$$

Let $\Sigma$ be the following set of wffs:

(a) $\neg(C_{n,x} \land C_{m,x})$ for $n \neq m \in \mathbb{N}^+$, $x \in S_n \cap S_m$.

(b) $\neg(C_{n,x} \land C_{n,y})$ for $n \in \mathbb{N}^+$, $x \neq y \in S_n \cap S_m$.

(c) $(C_{n,x_1} \lor \ldots \lor C_{n,x_k})$ for $n \in \mathbb{N}^+$, where $S_n = \{x_1, \ldots, x_k\}$.
Claim 14.7. Suppose that \( v \) is a truth assignment which satisfies \( \Sigma \). Then we can define a system of distinct representatives by

\[
x \in S_n \iff v(C_{n,x}) = T.
\]

Proof. By (b) and (c), each \( S_n \) gets assigned a unique representative. By (a), distinct sets \( S_m \neq S_m \) get assigned distinct representatives.

Next we prove that \( \Sigma \) is finitely satisfiable. So let \( \Sigma_0 \subseteq \Sigma \) be any finite subset. Let \( \{i_1, \ldots, i_l\} \) be the indices that are mentioned in \( \Sigma_0 \). Then \( \{S_{i_1}, \ldots S_{i_l}\} \) satisfies condition \( (H) \). By Hall’s Theorem, there exists a set of distinct representatives for \( \{S_{i_1}, \ldots S_{i_l}\} \); say, \( x_r \in S_{i_r} \). Let \( v_0 \) be the truth assignment such that for \( 1 \leq r \leq l \) and \( x \in S_{i_r} \),

\[
v(C_{i_r,x}) = T \iff x = x_r.
\]

Clearly \( v_0 \) satisfies \( \Sigma_0 \).

By the Compactness Theorem, \( \Sigma \) is satisfiable. Hence there exists a system of distinct representatives.

15 Proof of compactness

Theorem 15.1 (The Compactness Theorem). If \( \Sigma \) is a finitely satisfiable set of wffs, then \( \Sigma \) is satisfiable.

Basic idea Imagine that for each sentence symbol \( A_n \), either \( A_n \in \Sigma \) or \( \neg A_n \in \Sigma \). Then there is only one possibility for a truth assignment \( v \) which satisfies \( \Sigma \): namely,

\[
v(A_n) = T \iff A_n \in \Sigma.
\]

Presumably this \( v \) works...

In the general case, we extend \( \Sigma \) to a finitely satisfiable set \( \Delta \) as above. For technical reasons, we construct \( \Delta \) so that for every wff \( \alpha \), either \( \alpha \in \Delta \) or \( \neg \alpha \in \Delta \).

Lemma 15.2. Suppose that \( \Sigma \) is a finitely satisfiable set of wffs. If \( \alpha \) is any wff, then either \( \Sigma \cup \{\alpha\} \) is finitely satisfiable or \( \Sigma \cup \{\neg \alpha\} \) is finitely satisfiable.

Proof. Suppose that \( \Sigma \cup \{\alpha\} \) isn’t finitely satisfiable. Then there exists a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \cup \{\alpha\} \) isn’t satisfiable. Thus \( \Sigma \models \neg \alpha \). We claim that \( \Sigma \cup \neg \alpha \) is finitely satisfiable. Let \( \Delta \subseteq \Sigma \cup \{\neg \alpha\} \) be any finite subset. If \( \Delta \subseteq \Sigma \) then \( \Delta \) is satisfiable. Hence we can suppose that \( \Delta = \Delta_0 \cup \{\neg \alpha\} \) for some finite subset \( \Delta_0 \subseteq \Sigma \). Since \( \Sigma \) is finitely satisfiable, there exists a truth assignment \( v \) which satisfies \( \Sigma_0 \cap \Delta_0 \). Since \( \Sigma_0 \models \neg \alpha \), it follows that \( v(\neg \alpha) = T \). Hence \( v \) satisfies \( \Delta_0 \cup \{\neg \alpha\} \).
Proof of the Compactness Theorem. Let $\Sigma$ be a finitely satisfiable set of wffs. Let 

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots \ n \geq 1$$

be an enumeration of all the wffs $\alpha \in L$. We shall inductively define an increasing sequence of finitely satisfiable sets of wffs

$$\Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_n \subseteq \ldots$$

First let $\Delta_0 = \Sigma$. Suppose inductively that $\Delta_n$ has been defined. Then

$$\Delta_{n+1} = \begin{cases} 
\Delta_n \cup \{\alpha_{n+1}\}, & \text{if this is finitely satisfiable} \\
\Delta_n \cup \{\neg \alpha_{n+1}\}, & \text{otherwise}.
\end{cases}$$

By the lemma, $\Delta_{n+1}$ is also finitely satisfiable. Finally define

$$\Delta = \bigcup_n \Delta_n.$$

Claim 15.3. $\Delta$ is finitely satisfiable.

Proof. Suppose that $\Phi \subseteq \Delta$ is a finite subset. Then there exists an $n$ such that $\Phi \subseteq \Delta_n$. Since $\Delta_n$ is finitely satisfiable, $\Phi$ is satisfiable. \qed

Claim 15.4. If $\alpha$ is any wff, then either $\alpha \in \Delta$ or $\neg \alpha \in \Delta$.

Proof. There exists an $n \geq 1$ such that $\alpha = \alpha_n$. By construction, either $\alpha_n \in \Delta_{n+1}$ or $\neg \alpha_n \in \Delta_{n+1}$, and $\Delta_{n+1} \subseteq \Delta$. \qed

Define a truth assignment $\nu: L \rightarrow \{T, F\}$ by

$$\nu(A_t) = T \ \text{iff} \ A_t \in \Delta.$$ 

Claim 15.5. For every wff $\alpha$, $\nu(\alpha) = T$ iff $\alpha \in \Delta$.

Proof. We argue by induction on the length $m$ of the wff $\alpha$. First suppose that $m = 1$. Then $\alpha$ is a sentence symbol; say, $\alpha = A_t$. By definition

$$\nu(A_t) = \nu(A_t) = T \ \text{iff} \ A_t \in \Delta.$$ 

Now suppose that $m > 1$. Then $\alpha$ has the form

$$\neg \beta, (\beta \land \gamma), (\beta \lor \gamma), (\beta \rightarrow \gamma), (\beta \leftrightarrow \gamma)$$

for some shorter wffs $\beta, \gamma$. 

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Case 1 Suppose that $\alpha = (\neg \beta)$. Then

\[ \bar{v}(\alpha) = T \quad \text{iff} \quad \bar{v}(\beta) = F \]
\[ \text{iff} \quad \beta \notin \Delta \quad \text{by induction hypothesis} \]
\[ \text{iff} \quad (\neg \beta) \in \Delta \quad \text{by Claim 15.4} \]
\[ \text{iff} \quad \alpha \in \Delta \]

Case 2 Suppose that $\alpha$ is $(\beta \lor \gamma)$. First suppose that $\bar{v}(\alpha) = T$. Then $\bar{v}(\beta) = T$ or $\bar{v}(\gamma) = T$. By induction hypothesis, $\beta \in \Delta$ or $\gamma \in \Delta$. Since $\Delta$ is finitely satisfiable, $\{\beta, (\neg(\beta \lor \gamma))\} \notin \Delta$ and $\{\gamma, (\neg(\beta \lor \gamma))\} \notin \Delta$. Hence $(\neg(\beta \lor \gamma)) \notin \Delta$ and so $(\beta \lor \gamma) \in \Delta$.

Conversely suppose that $(\beta \lor \gamma) \in \Delta$. Since $\Delta$ is finitely satisfiable, $\{\neg \beta, (\neg \gamma), (\beta \lor \Gamma)\} \notin \Delta$. Hence $(\neg \beta) \notin \Delta$ or $(\neg \gamma) \notin \Delta$; and so $\beta \in \Delta$ or $\gamma \in \Delta$. By induction hypothesis, $\bar{v}(\beta) = T$ or $\bar{v}(\gamma) = T$. Hence $\bar{v}(\beta \lor \gamma) = T$.

Exercise 15.6. Write out the details for the other cases.

Thus $v$ satisfies $\Delta$. Since $\Sigma \subseteq \Delta$, it follows that $v$ satisfies $\Sigma$.\[\square\]