10 Propositional logic

“The study of how the truth value of compound statements depends on those of simple statements.”

A reminder of truth-tables.

and $\wedge$

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or $\vee$

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not $\neg$

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material implication $\rightarrow$

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iff $\leftrightarrow$

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Now our study actually begins... First we introduce our formal language.

Definition 10.1. The alphabet consists of the following symbols:
1. the sentence connectives
   \( \neg, \wedge, \vee, \rightarrow, \leftrightarrow \)

2. the punctuation symbols
   (, )

3. the sentence symbols
   \( A_1, A_2, \ldots, A_n, \ldots, n \geq 1 \)

**Remark 10.2.** Clearly the alphabet is countable.

**Definition 10.3.** An *expression* is a finite sequence of symbols from the alphabet.

**Example 10.4.** The following are expressions:
\[(A_1 \wedge A_2), ((\neg\rightarrow))A_3\]

**Remark 10.5.** Clearly the set of expressions is countable.

**Definition 10.6.** The set of *well-formed formulas* (wffs) is defined recursively as follows:

1. Every sentence symbol \( A_n \) is a wff.

2. If \( \alpha \) and \( \beta \) are wffs, then so are
   \( (\neg\alpha), (\alpha\wedge\beta), (\alpha\vee\beta), (\alpha\rightarrow\beta), (\alpha\leftrightarrow\beta) \)

3. No expression is a wff unless it is compelled to be so by repeated applications of (1) and (2).

**Remark 10.7.**

1. From now on we omit clause (3) in any further recursive definitions.

2. Clearly the set of wffs is countably infinite.

3. Because the definition of a wff is recursive, most of the properties of wffs are proved by induction on the length of a wff.

**Example 10.8.**

1. \( (A_1\rightarrow(\neg A_2)) \) is a wff.

2. \( ((A_1\wedge A_2) \) is not a wff. How can we prove this?

**Proposition 10.9.** *If \( \alpha \) is a wff, then \( \alpha \) has the same number of left and right parentheses.*
Proof. We argue by induction on the length $n \geq 1$ of the wff $\alpha$. First suppose that $n = 1$. Then $\alpha$ must be a sentence symbol, say $A_n$. Clearly the result holds in this case.

Now suppose that $n > 1$ and that the result holds for all wffs of length less than $n$. Then $\alpha$ must have one of the following forms:

$$(\neg \beta), (\beta \land \gamma), (\beta \lor \gamma), (\beta \rightarrow \gamma), (\beta \leftrightarrow \gamma)$$

for some wffs $\beta, \gamma$ of length less than $n$. By induction hypothesis the result holds for both $\beta$ and $\gamma$. Hence the result also holds for $\alpha$.

\[\square\]

**Definition 10.10.** $\mathcal{L}$ is the set of sentence symbols. $\bar{\mathcal{L}}$ is the set of wffs. $\{T, F\}$ is the set of truth values.

**Definition 10.11.** A *truth assignment* is a function $\nu: \mathcal{L} \rightarrow \{T, F\}$.

**Definition 10.12.** Let $\nu$ be a truth assignment. Then we define the extension $\bar{\nu}: \bar{\mathcal{L}} \rightarrow \{T, F\}$ recursively as follows.

0. If $A_n \in \mathcal{L}$ then $\bar{\nu}(A_n) = \nu(A_n)$.

For any $\alpha, \beta \in \bar{\mathcal{L}}$

1. $\bar{\nu}((\neg \alpha)) =$
   - $T$ if $\nu(\alpha) = F$
   - $F$ otherwise

2. $\bar{\nu}((\alpha \land \beta)) =$
   - $T$ if $\nu(\alpha) = \nu(\beta) = T$
   - $F$ otherwise

3. $\bar{\nu}((\alpha \lor \beta)) =$
   - $F$ if $\nu(\alpha) = \nu(\beta) = F$
   - $T$ otherwise

4. $\bar{\nu}((\alpha \rightarrow \beta)) =$
   - $F$ if $\nu(\alpha) = T$ and $\nu(\beta) = F$
   - $T$ otherwise

5. $\bar{\nu}((\alpha \leftrightarrow \beta)) =$
   - $T$ if $\nu(\alpha) = \nu(\beta)$
   - $F$ otherwise

**Possible problem.** Suppose there exists a wff $\alpha$ such that $\alpha$ has both the forms $(\beta \rightarrow \gamma)$ and $(\sigma \land \varphi)$ for some wffs $\beta, \gamma, \sigma, \varphi$. Then there will be two (possibly conflicting) clauses which define $\bar{\nu}(\alpha)$.

Fortunately no such $\alpha$ exists...

**Theorem 10.13 (Unique readability).** If $\alpha$ is a wff of length greater than 1, then there exists exactly one way of expressing $\alpha$ in the form:

$$(\neg \beta), (\beta \land \gamma), (\beta \lor \gamma), (\beta \rightarrow \gamma), \text{ or } (\beta \leftrightarrow \gamma)$$

for some shorter wffs $\beta, \gamma$. 
We shall make use of the following result.

**Lemma 10.14.** Any proper initial segment of a wff contains more left parentheses than right parentheses. Thus no proper initial segment of a wff is a wff.

*Proof.* We argue by induction on the length $n \geq 1$ of the wff $\alpha$. First suppose that $n = 1$. Then $\alpha$ is a sentence symbol, say $A_n$. Since $A_n$ has no proper initial segments, the result holds vacuously.

Now suppose that $n > 1$ and that the result holds of all wffs of length less than $n$. Then $\alpha$ must have the form 

$$(\neg \beta), (\beta \land \gamma), (\beta \lor \gamma), (\beta \rightarrow \gamma), \text{ or } (\beta \leftarrow \gamma)$$

for some shorter wffs $\beta$ and $\gamma$. By induction hypothesis, the result holds for both $\beta$ and $\gamma$. We just consider the case when $\alpha$ is $(\beta \land \gamma)$. (The other cases are similar.) The proper initial segments of $\alpha$ are:

1. $( \, )$

2. $(\beta_0$ where $\beta_0$ is an initial segment of $\beta$

3. $(\beta \land$

4. $(\beta \land \gamma_0$ where $\gamma_0$ is an initial segment of $\gamma$. 

Using the induction hypothesis and the previous proposition (Proposition 10.9), we see that the result also holds for $\alpha$.  

*Proof of Theorem 10.13.* Suppose, for example, that $\alpha = (\beta \land \gamma) = (\sigma \land \varphi)$. Deleting the first $( \, )$ we obtain that $\beta \land \gamma = \sigma \land \varphi$. 

Suppose that $\beta \neq \sigma$. Then wlog $\beta$ is a proper initial segment of $\sigma$. But then $\beta$ isn’t a wff, which is a contradiction. Hence $\beta = \sigma$. Deleting $\beta$ and $\sigma$, we obtain that $\land \gamma = \land \varphi$ and so $\gamma = \varphi$.

Next suppose that $\alpha = (\beta \land \gamma) = (\sigma \rightarrow \varphi)$.

Arguing as above, we find that $\beta = \sigma$ and so $\land \gamma = \rightarrow \varphi)$ which is a contradiction.

The other cases are similar.  

**Definition 10.15.** Let $v: \mathcal{L} \rightarrow \{T, F\}$ be a truth assignment.

1. If $\varphi$ is a wff, then $v$ satisfies $\varphi$ iff $v(\varphi) = T$.

2. If $\Sigma$ is a set of wffs, then $v$ satisfies $\Sigma$ iff $v(\sigma) = T$ for all $\sigma \in \Sigma$.  

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3. $\Sigma$ is *satisfiable* iff there exists a truth assignment $v$ which satisfies $\Sigma$.

**Example 10.16.** 1. Suppose that $v: L \to \{T, F\}$ is a truth assignment and that $v(A_1) = F$ and $v(A_2) = T$. Then $v$ satisfies $(A_1 \to A_2)$.

2. $\Sigma = \{A_1, (\neg A_2), (A_1 \to A_2)\}$ is *not* satisfiable.

**Exercise 10.17.** Suppose that $\varphi$ is a wff and $v_1, v_2$ are truth assignments which agree on all sentence symbols appearing in $\varphi$. Then $\bar{v}_1(\varphi) = \bar{v}_2(\varphi)$. (*Hint:* argue by induction on the length of $\varphi$.)

**Definition 10.18.** Let $\Sigma$ be a set of wffs and let $\varphi$ be a wff. Then $\Sigma$ *tautologically implies* $\varphi$, written $\Sigma \models \varphi$, iff every truth assignment which satisfies $\Sigma$ also satisfies $\varphi$.

**Important Observation.** Thus $\Sigma \models \varphi$ iff $\Sigma \cup \{\neg \varphi\}$ is not satisfiable.

**Example 10.19.** $\{A_1, (A_1 \to A_2)\} \models A_2$.

**Definition 10.20.** The wffs $\varphi, \psi$ are *tautologically equivalent* iff both $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

**Example 10.21.** $(A_1 \to A_2)$ and $((\neg A_2) \to (\neg A_1))$ are tautologaly equivalent.

**Exercise 10.22.** Let $\sigma, \tau$ be wffs. Then the following statements are equivalent.

1. $\sigma$ and $\tau$ are tautologically equivalent.

2. $(\sigma \leftrightarrow \tau)$ is a tautology.

(*Hint: do not* argue by induction on the lengths of the wffs.)