7 Binary relations

Definition 7.1. A binary relation on a set $A$ is a subset $R \subseteq A \times A$. We usually write $aRb$ instead of writing $(a, b) \in R$. 

Example 7.2. 1. The order relation on $\mathbb{N}$ is given by 
\[ \{ (n, m) \mid n, m \in \mathbb{N}, \; n < m \} . \]
2. The division relation $D$ on $\mathbb{N}\setminus\{0\}$ is given by 
\[ D = \{ (n, m) \mid n, m \in \mathbb{N}, \; n \text{ divides } m \} . \]

Observation Thus $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ is the collection of all binary relations on $\mathbb{N}$. Clearly $\mathcal{P}(\mathbb{N} \times \mathbb{N}) \sim \mathcal{P}(\mathbb{N})$ and so $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ is uncountable.

Definition 7.3. Let $R$ be a binary relation on $A$.

1. $R$ is reflexive iff $xRx$ for all $x \in A$.
2. $R$ is symmetric iff $xRy$ implies $yRx$ for all $x, y \in A$.
3. $R$ is transitive iff $xRy$ and $yRz$ implies $xRz$ for all $x, y, z \in A$.

$R$ is an equivalence relation iff $R$ is reflexive, symmetric, and transitive.

Example 7.4. Define the relation $R$ on $\mathbb{Z}$ by 
\[ aRb \text{ iff } 3|a - b. \]

Proposition 7.5. $R$ is an equivalence relation.

Exercise 7.6. Let $A = \{ (a, b) \mid a, b \in \mathbb{Z}, \; b \neq 0 \}$. Define the relation $S$ on $A$ by 
\[ (a, b)S(c, d) \text{ iff } ad - bc = 0. \]
Prove that $S$ is an equivalence relation.

Definition 7.7. Let $R$ be an equivalence relation on $A$. For each $x \in A$, the equivalence class of $x$ is 
\[ [x] = \{ y \in A \mid xRy \} . \]

Example 7.4 Cont. The distinct equivalence classes are 
\[ [0] = \{ \ldots, -6, -3, 0, 3, 6, \ldots \} \]
\[ [1] = \{ \ldots, -5, -2, 1, 4, 7, \ldots \} \]
\[ [2] = \{ \ldots, -4, -1, 2, 5, 8, \ldots \} \]

Definition 7.8. Let $A$ be a nonempty set. Then $\{ B_i \mid i \in I \}$ is a partition of $A$ iff the following conditions hold:
1. $\emptyset \neq B_i$ for all $i \in I$.

2. If $i \neq j \in I$, then $B_i \cap B_j = \emptyset$.

3. $A = \bigcup_{i \in I} B_i$.

**Theorem 7.9.** Let $R$ be an equivalence relation on $A$.

1. If $a \in A$ then $a \in [a]$.

2. If $a, b \in A$ and $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

Hence the set of distinct equivalence classes forms a partition of $A$.

**Proof.**

1. Let $a \in A$. Since $R$ is reflexive, $aRa$ and so $a \in [a]$.

2. Suppose that $c \in [a] \cap [b]$. Then $aRc$ and $bRc$. Since $R$ is symmetric, $cRb$. Since $R$ is transitive, $aRb$. We claim that $[b] \subseteq [a]$. To see this, suppose that $d \in [b]$. Then $bRd$. Since $aRb$ and $bRd$, it follows that $aRd$. Thus $d \in [a]$. Similarly, $[a] \subseteq [b]$ and so $[a] = [b]$.

**Theorem 7.10.** Let $\{B_i \mid i \in I\}$ be a partition of $A$. Define a binary relation $R$ on $A$ by

$$aRb \text{ iff there exists } i \in I \text{ such that } a, b \in B_i.$$ 

Then $R$ is an equivalence relation whose equivalence classes are precisely $\{B_i \mid i \in I\}$.

**Example 7.11.** How many equivalence relations can be defined on $A = \{1, 2, 3\}$?

**Sol’n** This is the same as asking how many partitions of $A$ exist.

$$\{\{1, 2, 3\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1\}, \{2\}, \{3\}\}$$

Hence there are 5 equivalence relations on $\{1, 2, 3\}$.

**Exercise 7.12.** How many equivalence relations can be defined on $A = \{1, 2, 3, 4\}$?

**Challenge** Let $\text{EQ}(\mathbb{N})$ be the collection of equivalence relations on $\mathbb{N}$. Prove that $\text{EQ}(\mathbb{N}) \sim \mathcal{P}(\mathbb{N})$. 

2006/02/06
8 Linear orders

Definition 8.1. Let $R$ be a binary relation on $A$.

1. $R$ is irreflexive iff $\langle a, a \rangle \notin R$ for all $a \in A$.

2. $R$ satisfies the trichotomy property iff for all $a, b \in A$, exactly one of the following holds:
   
   $$aRb, \quad a = b, \quad bRa.$$

$\langle A, R \rangle$ is a linear order iff $R$ is irreflexive, transitive, and satisfies the trichotomy property.

Example 8.2. Each of the following are linear orders.

1. $\langle \mathbb{N}, < \rangle$

2. $\langle \mathbb{N}, > \rangle$

3. $\langle \mathbb{Z}, < \rangle$

4. $\langle \mathbb{Q}, < \rangle$

5. $\langle \mathbb{R}, < \rangle$

Definition 8.3. Let $R$ be a binary relation on $A$. Then $\langle A, R \rangle$ is a partial order iff $R$ is irreflexive and transitive.

Example 8.4. Each of the follow are partial orders, but not linear orders.

1. Let $A$ be any nonempty set containing at least two elements. Then $\langle \mathcal{P}(A), \subseteq \rangle$ is a partial order.

2. Let $D$ be the divisability relation on $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Then $\langle \mathbb{N}^+, D \rangle$ is a partial order.

Definition 8.5. Let $\langle A, < \rangle$ and $\langle B, < \rangle$ be partial orders. A map $f: A \to B$ is an isomorphism iff the following conditions are satisfied.

1. $f$ is a bijection

2. For all $x, y \in A$, $x < y$ iff $f(x) < f(y)$.

In this case, we say that $\langle A, < \rangle$ and $\langle B, < \rangle$ are isomorphic and write $\langle A, < \rangle \cong \langle B, < \rangle$.

Example 8.6. $\langle \mathbb{Z}, < \rangle \cong \langle \mathbb{Z}, > \rangle$
Proof. Let \( f: \mathbb{Z} \to \mathbb{Z} \) be the map defined by \( f(x) = -x \). Clearly \( f \) is a bijection. Also, if \( x, y \in \mathbb{Z} \), then \( x < y \)
\[\text{iff } -x > -y\]
\[\text{iff } f(x) > f(y).\]
Thus \( f \) is an isomorphism.

Example 8.7. \( \langle \mathbb{N}, < \rangle \not\cong \langle \mathbb{Z}, < \rangle \).

Proof. Suppose that \( f: \mathbb{N} \to \mathbb{Z} \) is an isomorphism. Let \( f(0) = z \). Since \( f \) is a bijection, there exists \( n \in \mathbb{N} \) such that \( f(n) = z - 1 \). But then \( n > 0 \) and \( f(n) < f(0) \), which is a contradiction.

Exercise 8.8. Prove that \( \langle \mathbb{Z}, < \rangle \not\cong \langle \mathbb{Q}, < \rangle \).

Example 8.9. \( \langle \mathbb{Q}, < \rangle \not\cong \langle \mathbb{R}, < \rangle \).

Proof. Since \( \mathbb{Q} \) is countable and \( \mathbb{R} \) is uncountable, there does not exist a bijection \( f: \mathbb{Q} \to \mathbb{R} \). Hence there does not exist an isomorphism \( f: \mathbb{Q} \to \mathbb{R} \).

Example 8.10. \( \langle \mathbb{R}, < \rangle \not\cong \langle \mathbb{R} \setminus \{0\}, < \rangle \).

Proof. Suppose that \( f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is an isomorphism. For each \( n \geq 1 \), let \( r_n = f(1/n) \). Then
\[ r_1 > r_2 > \ldots > r_n > \ldots > f(-1). \]
Let \( s \) be the greatest lower bound of \( \{r_n \mid n \geq 1\} \). Then there exists \( t \in \mathbb{R} \setminus \{0\} \) such that \( f(t) = s \). Clearly \( t < 0 \). Hence \( f(t/2) > s \). But then there exists \( n \geq 1 \) such that \( r_n < f(t/2) \). But this means that \( t/2 < 1/n \) and \( f(t/2) > f(1/n) \), which is a contradiction.

Question 8.11. Is \( \langle \mathbb{Q}, < \rangle \cong \langle \mathbb{Q} \setminus \{0\}, < \rangle \)?

Definition 8.12. For each prime \( p \),
\[ \mathbb{Z}[1/p] = \{ a/p^n \mid a \in \mathbb{Z}, \ n \in \mathbb{N} \}. \]

Question 8.13. Is \( \langle \mathbb{Z}[1/2], < \rangle \cong \langle \mathbb{Z}[1/3], < \rangle \)?

Definition 8.14. A linear order \( \langle D, < \rangle \) is a dense linear order without endpoints or DLO iff the following conditions hold.

1. For all \( a, b \in D \), if \( a < b \), then there exists \( c \in D \) such that \( a < c < b \).
2. For all \( a \in D \), there exists \( b \in D \) such that \( a < b \).
3. For all \( a \in D \), there exists \( b \in D \) such that \( b < a \).

Example 8.15. The following are DLOs.
1. \( \langle \mathbb{Q}, < \rangle \)
2. \( \langle \mathbb{R}, < \rangle \)
3. \( \langle \mathbb{Q} \setminus \{0\}, < \rangle \)
4. \( \langle \mathbb{R} \setminus \{0\}, < \rangle \)

**Theorem 8.16.** For each prime \( p \), \( \langle \mathbb{Z}/p, < \rangle \) is a DLO.

**Proof.** Clearly \( \langle \mathbb{Z}/p, < \rangle \) linear order without endpoints. Hence it is enough to show that \( \mathbb{Z}/p \) is dense. Suppose that \( a, b \in \mathbb{Z}/p \). Then there exists \( c, d \in \mathbb{Z} \) and \( n \in \mathbb{N} \) such that \( a = c/p^n \) and \( b = d/p^n \). Clearly \( a < a + (1/p^n) \leq b \). Consider

\[
 r = \frac{c}{p^n} + \frac{1}{p^n} = \frac{cp^n + 1}{p^n} \in \mathbb{Z}/p.
\]

Then \( a < r < b \). □

**Theorem 8.17.** If \( \langle A, < \rangle \) and \( \langle B, < \rangle \) are countable DLOs then \( \langle A, < \rangle \cong \langle B, < \rangle \).

**Corollary 8.18.** \( \langle \mathbb{Q}, < \rangle \cong \langle \mathbb{Q} \setminus \{0\}, < \rangle \). □

**Corollary 8.19.** \( \langle \mathbb{Z}/2, < \rangle \cong \langle \mathbb{Z}/3, < \rangle \). □

**Corollary 8.20.** If \( p \) is any prime, then \( \langle \mathbb{Z}/p, < \rangle \cong \langle \mathbb{Q}, < \rangle \). □

**Proof of Theorem 8.17.** Let \( A = \{a_n \mid n \in \mathbb{N}\} \) and \( B = \{b_n \mid n \in \mathbb{N}\} \). First define \( A_0 = \{a_0\} \) and \( B_0 = \{b_0\} \) and let \( f_0 : A_0 \rightarrow B_0 \) be the map defined by \( f_0(a_0) = b_0 \).

Now suppose inductively that we have defined a function \( f_n : A_n \rightarrow B_n \) such that the following conditions are satisfied.

1. \( \{a_0, \ldots, a_n\} \subseteq A_n \subseteq A \).
2. \( \{b_0, \ldots, b_n\} \subseteq B_n \subseteq B \).
3. \( f_n : A_n \rightarrow B_n \) is an order preserving bijection.

We now extend \( f_n \) to a suitable function \( f_{n+1} \).

**Step 1** If \( a_{n+1} \in A_n \), then let \( A'_n = A_n \setminus \{a_{n+1}\} \), \( B'_n = B_n \), and \( f'_n = f_n \). Otherwise, suppose for example that

\[
c_0 < c_1 < \ldots < c_i < a_{n+1} < c_{i+1} < \ldots < c_m
\]

where \( A_n = \{c_0, \ldots, c_m\} \). Choose some element \( b \in B \) such that \( f_n(c_i) < b < f_n(c_{i+1}) \) and define

\[
 A'_n = A_n \cup \{a_{n+1}\} \\
 B'_n = B_n \cup \{b\} \\
 f'_n = f_n \cup \{(a_{n+1}, b)\}
\]
Step 2 If $b_{n+1} \in B'_n$, then let $A_{n+1} = A'_n$, $B_{n+1} = B'_n$, and $f_{n+1} = f'_n$. Otherwise, suppose for example that

$$d_0 < d_1 < \ldots < d_j < b_{n+1} < d_{j+1} < \ldots < d_t$$

where $B'_n = \{d_0, \ldots, d_t\}$. Choose some element $a \in A$ such that $(f'_n)^{-1}(d_j) < a < (f'_n)^{-1}(d_{j+1})$ and define

$$A_{n+1} = A'_n \cup \{a\}$$
$$B_{n+1} = B'_n \cup \{b_{n+1}\}$$
$$f_{n+1} = f'_n \cup \{(a, b_{n+1})\}.$$

Finally, let $f = \bigcup_{n \geq 0} f_n$. Then $f : A \to B$ is an isomorphism.

9 Extensions

Definition 9.1. Suppose that $R, S$ are binary relations on $A$. Then $S$ extends $R$ iff $R \subseteq S$.

Example 9.2. Consider the binary relations $R, S$ on $\mathbb{N}$ defined by

$$R = \{(n, m) \mid n < m\}$$
$$S = \{(n, m) \mid n \leq m\}$$

Then $S$ extends $R$.

Example 9.3. Consider the partial order $\prec$ on $\{a, b, c, d, e\}$ which is

$$\{(d, b), (d, a), (d, e), (d, c), (a, b), (e, b), (c, b)\}.$$

Then we can extend $\prec$ to the linear order $<$ defined by the transitive closure of

$$d < e < c < a < b.$$

Exercise 9.4. If $\langle A, \prec \rangle$ is a finite partial order, then we can extend $\prec$ to a linear ordering $<$ of $A$.

Question 9.5. Does the analogous result hold if $\langle A, \prec \rangle$ is a infinite partial order?

Definition 9.6. If $A$ is a set and $n \geq 1$, then

$$A^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A\}.$$  

An $n$-ary relation on $A$ is a subset $R \subseteq A^n$.

An $n$-ary operation on $A$ is a function $f : A^n \to A$.  

2006/02/06