Homework-1 (Algebraic Gems)

Due date: October 10 (in class)

Collaboration is encouraged. However, the writeup should be your own.

1. Reverse Odd-town: There is a town with \( n \) people. The people like forming clubs. The town also has some rules. Every club must have an even number of people and the intersection of every pair of distinct clubs must have an odd number of people. What is the maximum number of clubs that can be formed?

2. There is a town with \( n \) people. The people like forming clubs. This town also has some rather strange rules. There is an integer \( k \) so that the intersection of every two distinct clubs must have exactly \( k \) members. Show that there cannot be more than \( n \) clubs that are formed.
   (Hint: consider vector space over the field of real numbers.)

3. Suppose \( P \) is a set of \( n \) points in the two dimensional Euclidean plane such that not all the points are on one line. For every pair of points in \( P \), one can consider the line through them. Then show that pairs of points from \( P \) define at least \( n \) distinct lines
   (Hint: Use the answer to the previous problem.)

4. Let \( G = (V, E) \) be a \( d \)-regular graph. Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) be the eigenvalues of its adjacency matrix. We showed in class that \( \lambda_1 = d \). Show that \( \lambda_n = -d \) if and only if the graph is bipartite.
   (Recall \( G \) is bipartite if its vertex set can be divided into two sets \( A \) and \( B \) so that all edges of \( G \) go between the sets \( A \) and \( B \).)

5. Fast multipoint evaluation

We showed using FFT how to multiply two polynomials of degree \( n \) in \( O(n \log n) \) time and how to divide two such polynomials in \( O(n \log^2 n) \) time. Using these two results, show how to evaluate a polynomial of degree \( n \) at \( n \) arbitrary points in \( O(n \cdot \text{poly}(\log n)) \) time. Outlined below are the steps of the proof.

Given the coefficient representation of a polynomial \( A \) and \( n \) points \( x_0, x_1, \ldots, x_{n-1} \), we wish to compute the \( n \) values \( A(x_0), A(x_1), \ldots, A(x_{n-1}) \). For \( 0 \leq i \leq j \leq n - 1 \), define the polynomials \( P_{ij} = \prod_{k=i}^{j}(x - x_k) \), and \( Q_{ij}(x) = A(x) \mod P_{ij}(x) \). Note that \( Q_{ij}(x) \) has degree at most \( j - i \).

(a) Prove that \( A(x) \mod (x - z) = A(z) \) for any point \( z \).
(b) Prove that $Q_{kk}(x) = A(x_k)$ and that $Q_{0,n-1}(x) = A(x)$. 

(c) Prove that for $i \leq k \leq j$, we have $Q_{ik}(x) = Q_{ij}(x) \mod P_{ik}(x)$ and $Q_{kj}(x) = Q_{ij}(x) \mod P_{kj}(x)$. 

(d) Give an $O(n \cdot \text{poly}(\log n))$-time algorithm to evaluate $A(x_0), A(x_1), ..., A(x_{n-1})$

(Hint: Use divide and conquer - to evaluate $A$ at all $n$ points, it suffices to evaluate $Q_{0,\frac{n}{2}-1}$ at the first $n/2$ points and $Q_{\frac{n}{2},n-1}$ at the last $n/2$ points.)