Homework-4 (Combinatorics 2)

Due date: April 27 (in class)

Collaboration is encouraged. However the writeup should be your own. If you use any resources such as books or the internet, you should make sure you mention this in the homework. If you collaborate with other students then make sure you acknowledge your collaborators. Some of the questions asked might be “classic” and hence their solutions might be found on the internet or in research papers. However, you should not refer to such sources.

1. Let $X, Y$ and $Z$ be three random variables. For $H$ being the binary entropy function, prove that $H(X | Y, Z) \leq H(X | Y)$.

2. Let $G = (V, E)$ be an undirected graph. Let $n_a$ be the number of cliques of size $a$ in $G$, and let $n_b$ be the number of cliques of size $b$ in $G$. Suppose that $a < b$. Then show that $(b! \cdot n_b)^a \leq (a! \cdot n_a)^b$.

3. For a graph $G$ on $2n$ vertices, we say that a subset of its edges forms a perfect matching, if the subset is of size $n$ and touches each vertex of $G$ exactly once. We say that a family of graphs $\mathcal{F}$ on the vertex set $[2n]$ is pm-intersecting, if for every $G_1, G_2 \in \mathcal{F}$, the edges of $G_1 \cap G_2$ contain a perfect matching. Prove that $|\mathcal{F}| \leq 2^{2n} - n$. Prove also that this bound is tight.

   (Hint: Try to come up with a graph with about $2n - 1$ edges such that the intersection of every pair of graphs from $\mathcal{F}$ must contain an edge from that graph.)

4. Let $K_n$ denote the complete bipartite graph on $n$ vertices. Suppose that $G_1, G_2, \ldots G_k$ are bipartite graphs on vertex set $[n]$, such that the union of their edges covers all the edges of a $K_n$ on vertex set $[n]$. Then show that $k \geq \log n$. 

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