Homework-3 (Combinatorics 2)

Due date: April 13 (in class)

Collaboration is encouraged. However the writeup should be your own. If you use any resources such as books or the internet, you should make sure you mention this in the homework. If you collaborate with other students then make sure you acknowledge your collaborators. Some of the questions asked might be “classic” and hence their solutions might be found on the internet or in research papers. However, you should not refer to such sources.

1. Prove the following regularity lemma:

Define an interval to be a set of consecutive integers. For real numbers $\alpha, \beta$ where $\alpha < \beta$, we let $(\alpha, \beta]$ denote the set of integers $r$ such that $\alpha < r \leq \beta$.

Let $S$ be a set of integers in $[n]$. For an interval $I \subseteq [n]$, we define the density of $I$ with respect to $S$ to be $d(I) = |S \cap I|/|I|$

For any interval $I \subseteq [n]$, we say that the interval is $\epsilon$-regular with respect to $S$, if for all subintervals $J \subseteq I$, such that $J \geq \epsilon|I|$, $|d(J) - d(I)| \leq \epsilon$.

We say that a partition is an $\epsilon$-regular equipartition of $[n]$ with $k$ parts, if all but $\epsilon \cdot k$ of the following intervals are $\epsilon$-regular with respect to $S$:

$(0, n/k], (n/k, 2n/k], (2n/k, 3n/k], \ldots, ((k-1)n/k, n]$

Prove that for every $\epsilon > 0$ and $t > 0$, there exists $T$ and $N$ such that for all $n > N$, suppose $S \subseteq [n]$ then $[n]$ has an $\epsilon$-regular equipartition wrt $S$, with $k$ parts, where $t \leq k \leq T$.

What kind of bound do you obtain for $T$ (in terms of $\epsilon$ and $t$)?

2. Prove the counting lemma for $K_4$. In particular, for every $\delta > 0$, show that there exist constants $c > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$, if $R(\Pi)$ denotes the $((\epsilon, \delta)$ reduced graph of a graph $G$ with $n$ vertices (and $n$ is large enough), and if there is a copy of $K_4$ in $R(\Pi)$, then there is a copy of $K_4$ in $G$. If each part of $R(\Pi)$ has at least $k$ vertices then show that the number of copies of $K_4$ in $G$ is at least $ck^4$.

(For those of you who want to try, try to prove the counting lemma for general graphs $H$. Also, try to do this independently without looking at reference material.)