Chapter 16

The Theory of Optimal Control and the Calculus of Variations†

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1. Background

"System theory" today connotes a loose collection of problems and methods held together by a central theme: to understand better the complex systems created by modern technology. Aside from certain combinatorial questions, most of present system theory is concerned with problems in automatic control and in statistical estimation and prediction, with emphasis on solutions that are optimal in some sense. These problems are attacked by a variety of ad hoc methods.

Recent research has shown how to formulate and resolve these problems in the spirit of the classical calculus of variations. This provides a unifying point of view. Eventually it should be possible to organize system theory as a rigorous and well-defined discipline. One example of this trend is the author's duality principle (see [1], [2], [3]) relating control and estimation. Conversely, problems in system theory are stimulating further research in the calculus of variations.

Let us look first at the historical background of the hamiltonian formulation of the calculus of variations. There is a long stream of scientific thought concerned with wave propagation and variational principles in physics. It begins with Huygens, continues with the work of John Bernoulli, and receives maturity at the hands of the great masters of the nineteenth century: Hamilton, Jacobi, and Lie. The most articulate representative of this tradition in recent times was C. Carathéodory (1873–1950). Beginning with his famous dissertation

† This research was sponsored by the United States Air Force under Contracts AF 49(638)-382 and AF 33(616)-6952.
of 1904, Carathéodory insisted on the hamiltonian point of view in the
calculus of variations throughout his lifetime. The evolution of his
thinking on this subject is carefully integrated in his last major work
[4]—a book that is hard to obtain and difficult to digest.

The theory of optimal control, under the assumption that the equa-
tions of motion are known exactly and the state can be measured in-
stantaneously, may be regarded as a generalization of the problem of
Lagrange in the calculus of variations; minimization of an integral sub-
ject to side conditions, which may be ordinary or differential equations.
Carathéodory’s work on the problem of Lagrange is incomplete, con-
sisting of only two papers [5], [6]; these papers are discussed briefly in
Chapter 18 of [4]. The problem is one of extreme difficulty and it has
received little attention until quite recently.

In [7] the present writer gave a new formulation of the problem of
optimal control from the hamiltonian point of view. The purpose of
this chapter is to extend this approach. We shall see that this formula-
tion—which differs from Carathéodory’s in essential details—explains
a number of recent results in the theory of control and provides a very
general framework for further research.

2. The Variational Problem in the Theory of Control

We assume that the control object is a dynamical system governed by
the differential equation

$$\frac{dx}{dt} = \dot{x} = f(x, u(t), t). \tag{2.1}$$

Here $x$ is a real $n$-vector, called the state of the system; $u(t)$ is a real
$m$-vector for each $t$; $f$ is a real $n$-vector that is continuously differen-
tiable in all arguments.

To avoid the cumbersome phrase “the state $x$ at time $t,”$ we shall
refer to the couple $(x, t)$ as a phase. The phase space is thus the cartes-
ian product of the state space $X \ (= \mathbb{R}^n)$ with the space $T \ (= \mathbb{R}^t)$ of all
values of the time.

We call the function $u(t)$ in (2.1) an admissible control if (a) it is piece-
wise continuous in $t$; (b) for each $t$, its values belong to a given closed
subset $U(t)$ of $\mathbb{R}^m$.

For any admissible control $u$ and any initial phase $(x_0, t_0)$, there
exists a unique absolutely continuous function $\phi$ of $t$, denoted by

$$\phi(t) = \phi_u(t; x_0, t_0),$$
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that satisfies (2.1) almost everywhere and has the property
\[ \phi(t_0) = \phi_u(t_0; x_0, t_0) = x_0. \]

We call \( \phi_u(t; x_0, t_0) \) the motion of (2.1) passing through \( x_0 \) at time \( t_0 \) under the action of the control \( u \). Sometimes we shall write \( x(t) = \phi(t) \) to emphasize the fact that the value of \( \phi \) at some fixed \( t \) is the state of the system at that time.

We call \( x^* \) an equilibrium state if there is some control \( u^* \) such that \( \phi_u(t; x^*, t_0) = x^* \) for all \( t, t_0 \), or, equivalently, if \( f(x^*, u^*(t), t) = 0 \).

To state the control problem in its simplest form, it is assumed further that physical measurements are available giving the exact numerical value of the state at every instant of time, though of course this is a gross idealization from the engineering point of view. We want to determine \( u(t) \) as a function of \( x(t) \) so that motions of (2.1) have certain extremal properties. The expression of \( u(t) \) as a function of \( x(t) \) is commonly called feedback in engineering. We denote this functional relationship by
\[ u(t) = k(x(t), t), \tag{2.2} \]
and refer to the function \( k \) as the control law. A control law is admissible if \( k(x, t) \in U(t) \) for all \( t \).

Let \( (x_0, t_0) \) be an arbitrary phase and let \( S \) be a surface in the phase space. Consider the following scalar functional of motions of (2.1):
\[ V(x_0, t_0, S; u) = \lambda(\phi_u(t_1; x_0, t_0), t_1) + \int_{t_0}^{t_1} L(\phi_u(t; x_0, t_0), u(t), t) \, dt, \tag{2.3} \]
where \( L, \lambda \) are scalar functions and \( t_1 \) is the first instant of time after \( t_0 \) when the motion enters the set \( S \). Thus it suffices for \( \lambda \) to be defined only on \( S \). We call \( t_1 \) the terminal time and assume that \( L, \lambda \) are continuously differentiable in all arguments.

In terms of these notations, we can now state the

**Optimal Control Problem.** Given any initial phase \((x_0, t_0)\), find a corresponding admissible control \( u \) defined in the interval \([t_0, t_1]\) at which the functional (2.3) assumes its infimum (or supremum) with regard to the set of all admissible controls.

Actually, for technological reasons one usually sets a slightly stronger objective.
**Optimal Feedback Control Problem.** Find a control law such that when (2.2) is substituted in (2.1) the functional (2.3) assumes its infimum (or supremum) with regard to the set of all admissible control laws.

Bellman's principle of optimality shows that we can always define an optimal control law along every optimal motion. Hence the two foregoing problems are abstractly equivalent.

If Equation (2.1) depends on stochastic factors, however, then the infimum of (2.3) with respect to all admissible control laws will usually be lower than with respect to all admissible controls that are uniquely determined by the initial phase. This is owing to the fact that the control law takes into account not only the initial state but successive states as well. The added information so obtained may result in a better optimum.

Before embarking on a detailed analysis of the control problem, let us mention a number of typical examples that may be put into this formulation.

*Terminal Control*

The problem is to bring the state of the system as close as possible to a given terminal state $x_1$ at a given terminal time $t_1$. Then $L = 0$, $\lambda(x)$ is the distance of $x$ from $x_1$, and $S = X \times \{t_1\}$.

*Minimal-Time Control*

Suppose we want to reach a state $x_1$ from $(x_0, t_0)$ in the shortest possible time. We then set $L = 1$, $\lambda = 0$, and $S = \{x_1\} \times T$. This problem ordinarily has a solution only if $U(t)$ is a bounded set for all $t \geq t_0$.

*Regulator Problem*

We assume that the system is in some initial phase $(x_0, t_0)$ and we wish to return to an equilibrium state $x^*$ in such a way that some integral of the motion is minimized. We then usually take $L$ and $\lambda$ as non-negative. The dependence of $L$ on $u$ is needed because otherwise the problem may not have a solution. The set $S$ is again $X \times \{t_1\}$.

*Pursuit Problem*

We are given a moving target $\xi(t)$. The problem is to bring the motion to phase $(\xi(t), t)$ as soon as possible. This is a generalization of the minimal-time problem; we take $S = \{(\xi(t), t) ; t \in T\}$. 
Servomechanism Problem

This is a generalization of the regulator problem. We are given a desired state $\xi(t)$, $t \in T$. The problem is to cause the phase of the controlled motion to be as close as possible to $(\xi(t), t)$ on the interval $[t_0, t_1]$. The instantaneous distance between $(x(t), t)$ and $(\xi(t), t)$ is measured by $L$. The set $S$ is again as in the regulator problem, above.

Minimum Energy Control

We wish to transfer from an initial phase $(x_0, t_0)$ to a final phase $(x_i, t_i)$ with the expenditure of a minimal amount of control energy. In this case we take $L$ to be a nonnegative function of $u$, independent of $\phi$; $S$ is the set consisting of the single point $(x_i, t_i)$; $\lambda$ is immaterial.

Isoperimetric Problems

Suppose that the optimal motions must satisfy also the so-called isoperimetric constraints

$$\int_{t_0}^{t_1} f_{n+k} (\phi_u(t; x_0, t_0), u(t), t) \, dt \leq \alpha_k, \quad k = 1, \ldots, N - n. \quad (2.4)$$

These problems reduce immediately to the preceding ones when we replace the $n$-vector $x$ by an $N$-vector of which the last $N - n$ components satisfy the differential equations

$$\frac{dx_{n+k}}{dt} = f_{n+k} (x, u(t), t), \quad k = 1, \ldots, N - n; \quad (2.5)$$

the initial values are $x_{n+k}(t_0) = 0$, and the final values $x_{n+k}(t_i)$ are to lie on a surface $S$ for which $x_{n+k} \leq \alpha_k$.

3. Relations with the Calculus of Variations

The classical problem of Lagrange in the calculus of variations is concerned with the minimization of the integral

$$\int L(x(t), \dot{x}(t), t) \, dt \quad (3.1)$$

with respect to any smooth curve $x(t)$ that (a) connects a given point $(x_0, t_0)$ with a point $(x_i, t_i)$ lying on a given surface $S$, and (b) satisfies the constraints

$$g_i(x(t), \dot{x}(t), t) \equiv 0, \quad i = 1, \ldots, n - m. \quad (3.2)$$
There are two ways in which the optimal control problem discussed above differs from the problem of Lagrange. First, the function $L$ depends on $u$ rather than on $\dot{x}$; second, the constraints are of a mixed type:

$$\dot{x} - f(x, u(t), t) = 0 \quad \text{and} \quad u(t) \in U(t). \quad (3.3)$$

Neither of these differences is essential; inequality constraints such as $\alpha \geq 0$ can be replaced by equality constraints such as $\beta(\alpha) = 0$, where $\beta$ is a smooth function that is zero if $\alpha \geq 0$ and positive otherwise. Similarly, one can always express $u(t)$ by (3.3) as a function of $x, \dot{x}, t$ by introducing, if necessary, additional equality constraints. Hence the optimal control problem is formally identical with the problem of Lagrange, though the transformations necessary to establish the equivalence will be usually rather complicated. Moreover, because of difficulties arising from an explicit treatment of the constraints (3.2), the theory of the Lagrange problem today is far from adequate.

We therefore prefer to treat directly the problem of minimizing (2.3), subject to the constraints (3.3). This treatment includes the ordinary problem of the calculus of variations, obtained by setting $f(x, u, t) = u$ and $U(t) = R^n$, as well as the Lagrange problem after suitable transformation of the type just discussed.

Using the hamiltonian point of view, we do not need to transform the constraints (3.3) but can treat them directly. The principal idea is the following. We define a hamiltonian function not with the aid of the Legendre transformation, as is usual, but in a more general procedure by means of the so-called minimum principle. In this way the optimal control problem can be reduced to the solution of the Hamilton–Jacobi partial differential equation. The existence of a solution of the Hamilton–Jacobi equation is a sufficient condition for the solution $V^0(x, t)$ of the optimal control problem. If the function $V^0(x, t)$ is smooth, this condition is also necessary.

Unfortunately, quite often $V^0(x, t)$ does not have continuous partial derivatives with respect to $x$. In that case one cannot state necessary and sufficient conditions solely in terms of differential equations. But this is hardly the issue. Early in his career, Carathéodory took the following position:

The distinction between necessary and sufficient conditions seems, however, a little artificial; explicit proof that certain conditions are necessary is of interest only in cases in which one cannot resolve a problem at once, and it serves, above all, to limit the scope for future investigations. When, on the other hand, one has a solution possessing all the properties required by the theorem, it suffices to show that this solution is unique in order to have at
the same time the proof that all the conditions that serve to determine the solution are necessary.†

It has unfortunately become very common in physical and engineering applications to regard the extremals supplied by the Euler equations as the "solution" of a variational problem. There are two long-standing objections to this: (a) the Euler equations may not exist, as when $L$ is not sufficiently smooth; (b) the solution of the Euler equations may cease to define a minimum or maximum after a certain interval of time, as when the extremal contains conjugate points. The hamiltonian point of view, which aims to obtain sufficient conditions, avoids such difficulties at the outset by considering only those initial phases that can be connected by optimal motion with a phase on $S$, and by regarding the function $V^0 = \min V$ as abstractly defined in advance.

The dynamic programming method of Bellman proceeds from the same fundamental idea, differing only in detail from the hamiltonian methods. For a nonrigorous but highly enlightening discussion of the relations between the two, see the recent paper of Dreyfus [10].

4. The Hamilton–Jacobi Equation; Minimum Principle

Let us first obtain the sufficient condition. The starting point is the following trivial, well-known, but important observation concerning the optimal control problem.

**Lemma of Carathéodory** [4, p. 198]. Suppose there is a function $k(x, t)$ continuously differentiable in both its arguments and such that, for all $(x, t)$ in some region $G$ of the phase space,

i. $k(x, t) \subseteq U(t),$

ii. $L(x, k, t) = 0,$

iii. $L(x, u, t) > 0$ if $u \neq k(x, t).$

Consider motions of (2.1) with control law defined by (2.2), that is,

$$\frac{dx}{dt} = f(x, k(x, t), t).$$

(4.2)

Let the initial phase $(x_0, t_0)$ belong to $G$. Let $\lambda(x, t)$ be identically zero on some surface $S \subseteq G$ of the phase space. Then the following properties hold for any motion $\phi^0$ of (4.2) that connects $(x_0, t_0)$ with a phase on $S$ and remains entirely in $G$:

† Author's translation from French; author's italics. See [9], Introduction.
a. The value of the integral (2.3) is zero.

b. The motion \( \phi^0 \) provides the absolute minimum of (2.3) with respect to any other motion of (2.1) which connects \((x_0, t_0)\) with \(S\) and remains entirely in \(G\).

In short, the hypotheses of the lemma mean that at every point in \(G\) the integrand \(L\) has a unique, absolute minimum \(u^0 = k(x, t)\) with respect to all \(u\) satisfying the constraint (3.3). Then \(k\) is the unique optimal control law, and the optimal feedback problem is also solved.

Proof. Conclusion (a) is immediate, since for any motion \(\phi^0\) of (4.2) the integral (2.4) is zero by hypothesis (ii). Now let \(\phi^1\) be any other motion of (2.1) that connects \((x_0, t_0)\) with \(S\) without leaving \(G\), and for which \(V = 0\). Then by hypothesis (iii) and the continuity of \(L\), it is clear that along \(\phi^1\) we must have \(u^1(t) = k(\phi^1(t; x_0, t_0), t)\) at every continuity point of \(u^1(t)\), since otherwise we would have \(V > 0\). We would obtain the same motion if we let \(u^1(t)\) always be defined by this relation, that is, \(\phi^1(t; x_0, t_0) = \phi^1(t; x_0, t_0)\). But since \(k\) is continuously differentiable in \(x\), (4.2) defines a unique motion, and the proof of (b) is complete.

It should be noted that there may be phases in \(G\) such that the motion defined by (4.2) going through these phases is not optimal. This is owing to the possibility that a motion may leave \(G\) prior to reaching \(S\).

Now we try to construct a Lagrange function \(L^*\) and a corresponding function \(k\) satisfying the requirements of the lemma.

Suppose \(V^0(x, t)\) is a scalar function that is twice continuously differentiable in both arguments. Then\(^\dagger\)

\[
\int_{t_0}^{t_1} \left[ V^0(x, t) + f(x(u(t), t), x) \cdot V^0_{\tau}(x, t) \right] dt = V^0(x_1, t_1) - V^0(x_0, t_0) \tag{4.3}
\]

along any motion of (2.1) connecting the phase \((x_0, t_0)\) with the phase \((x_1, t_1)\) on \(S\). If we let

\[
V^0(x, t) = \lambda(x, t) \tag{4.4}
\]
on \(S\), then the optimal control problem obtained by replacing \(\lambda\) with \(\lambda^* = 0\) and \(L\) with

\[
L^*(x, u, t) = L(x, u, t) + V^0(x, t) + f(x(u, t), x) \cdot V^0_{\tau}(x, t) \tag{4.5}
\]
will be equivalent to the original problem, because the values of \(V\) and

\(^\dagger\) The dot denotes the inner product; \(V_i = \partial V/\partial t, V_x = \text{grad } V\).
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\( V^* \) will differ only by \( V^0(x_0, t_0) \), which does not depend on the control \( u \).

Let \( p \) be a real \( n \)-vector, called the \textit{costate}.

We define a scalar function \( H \) by

\[
H(x, p, t, u) = L(x, u, t) + f(x, u, t) \cdot p.
\]

We assume that \( H \) has a unique \textit{absolute} minimum for each \( t \) with respect to \( u(t) \in U(t) \) at the point

\[
u^0(t) = c(k, p, t);
\]

moreover, \( c \) is continuously differentiable in all arguments.

The scalar function \( H^0 \), defined by

\[
H^0(x, p, t) = \min_{u(t) \in U(t)} H(x, p, t, u)
\]

\[
= L(x, c(x, p, t), t) + f(x, c(x, p, t), t) \cdot p,
\]

is the \textit{hamiltonian} of the problem.

Finally, we assume that \( V^0(x, t) \) satisfies the Hamilton–Jacobi partial differential equation

\[
V^0_t + H^0(x, V^0_x, t) = 0
\]

with the boundary condition (4.4).

If these assumptions hold, we let

\[
p = V^0(x, t).
\]

Then

\[
L^*(x, u, t) = V^0_t(x, t) + H^0(x, V^0_x(x, t), t)
\]

will clearly satisfy the hypotheses of the lemma of Carathéodory, with \( k \) defined by

\[
k(x, t) = c(x, V^0_x(x, t), t).
\]

Moreover, by the lemma, we also have

\[
V^0(x_0, t_0) = \lambda(\phi^0(t_1), t_1) + \int_{t_0}^{t_1} L(\phi^0(t), k(\phi^0(t), t), t) \, dt
\]

along motions \( \phi^0 \) satisfying (4.2). In other words,

\[
V^0(x_0, t_0) = \min_u V(x_0, t_0, S; u)
\]

is the absolute minimum of the integral (2.3) with respect to admissible
controls; (4.12) is the optimal control law, and we have also solved the optimal feedback control problem.

Hence we have established:

**Theorem 4.1. Sufficient Condition.** Let \( H^0 \) be the absolute minimum of \( H = L + f \cdot p \) with respect to \( u(t) \in U(t) \). Suppose that \( u^0 = c \) is unique, that the differentiability hypotheses hold, that \( V^0 \) satisfies the Hamilton–Jacobi partial differential equation \( V^0_t + H^0(x, V^0_x, t) = 0 \) in a region \( G \subset S \), and that furthermore \( V = \lambda \) on \( S \). Then the following properties hold:

a. The function \( V^0(x_0, t_0) \) is the absolute minimum of (2.3) with respect to all motions which connect \( (x_0, t_0) \) with a phase on \( S \) without leaving \( G \).

b. The optimal control law is given by (4.12); with this control law any motion that eventually reaches \( S \) without leaving \( G \) is optimal.

The introduction of the hamiltonian function \( H^0 \) reduces the problem to one of ordinary minimization, which defines the optimal value of \( u(t) \) at each moment through (4.7). To achieve this, we bring in the auxiliary variable \( p \). To make sure that the point-by-point optimization based on \( p \) is consistent, we eliminate \( p \) by (4.10); the construction succeeds whenever \( V^0 \) is a solution of the Hamilton–Jacobi equation defined by \( H^0 \).

It is easy to prove the converse result—in other words, that the optimal value \( V^0 \) of (2.3) must satisfy the Hamilton–Jacobi equation—provided \( V^0 \) is a sufficiently smooth function of \( x \).

**Theorem 4.2. Necessary Condition.** Let \( G \) be a region in the phase space possessing the following properties:

i. There is an optimal motion from every phase in \( G \) to a phase on \( S \) that never leaves \( G \).

ii. The minimum value of (2.3), denoted by \( V^0(x, t) \), is twice continuously differentiable in both arguments.

iii. Every point in \( G \) that is not also on \( S \) has a neighborhood lying entirely in \( G \).

iv. \( \dagger \) For every phase in \( G \), \( H(x, V^0_x, t, u) \) given by (4.6) has an absolute minimum \( H^0(x, V^0_x, t) \) at \( u^0 = k(x, t) \) with respect to \( u(t) \in U(t) \).

v. \( \dagger \) The function \( k \) defining the minimum is differentiable in \( x \) and continuous in \( t \).

Then the function \( V^0(x, t) \) satisfies the Hamilton–Jacobi equation \( V^0_t + H^0(x, V^0_x, t) = 0 \) in the region \( G \).

\( \dagger \) These conditions may be checked from the given form of \( L, f, \) and \( U \), that is, without solving the variational problem.
Proof. Let \((x_0, t_0)\) be a phase in \(G\) for which the theorem is false. There are then two possibilities. We consider first

\[ V^0(x_0, t_0) + H^0(x_0, V^0_t(x_0, t_0), t_0) > 0. \] (4.15)

Let \(N \subset G\) be an open neighborhood of \((x_0, t_0)\) that is small enough that the inequality (4.15) remains true everywhere in \(N\). It is clear that \(N\) exists because of (iii) and because the left-hand side of (4.15) is continuous in \(x\) and \(t\).

Let \(\phi^0(t)\) be an optimal motion originating at \((x_0, t_0)\), and let \(u^0(t)\) be the corresponding optimal control. Then, because of the definition of \(H^0\), for all \(t\) such that \((\phi^0(t), t) \in N\) we have

\[ H(\phi^0(t), V^0_t(\phi^0(t), t, u^0(t)), t, u^0(t)) \geq H^0(\phi^0(t), V^0_t(\phi^0(t), t, u^0(t)), t). \] (4.16)

Combining (4.15) and (4.16), we obtain

\[-V^0_\partial(\phi^0(t), t) - V^0_\partial(\phi^0(t), t) \cdot f(\phi^0(t), u^0(t), t) \leq L(\phi^0(t), u^0(t), t) - \epsilon(t),\] (4.17)

with \(\epsilon > 0\) as long as \((\phi^0(t), t)\) remains in \(N\). Let \(t_1 > t_0\) such that \((\phi^0(t), t) \in N\) for all \(t \in [t_0, t_1]\). Integrating both sides of (4.17), we get

\[ V^0(x_0, t_0) - V^0(x_1, t_1) = \int_{t_0}^{t_1} [L(\phi^0(t), u^0(t), t) - \epsilon(t)] dt \]

or, using the definition of \(V^0(x_1, t_1)\) and letting \((x_2, t_2)\) be the phase on \(S\) reached by an optimal motion \(\phi^0\) starting at \((x_1, t_1)\),

\[ V^0(x_0, t_0) < \lambda(x_2, t_2) + \int_{t_0}^{t_1} L(\phi^0(t), u^0(t), t) \ dt, \]

which contradicts the assumption that \(\phi^0\) is optimal.

Now we suppose that \(N\) is an open neighborhood of \((x_0, t_0)\) throughout which the inequality (4.15) holds in the opposite sense. Then, by the definition of \(H^0\), we have

\[-V^0_\partial(x, t) - V^0_\partial(x, t) \cdot f(x, k(x, t), t) = L(x, k(x, t), t) + \epsilon(x, t), \]

where \(\epsilon > 0\) throughout \(N\).

Hence, integrating along the unique motion \(\phi_k(t; x_0, t_0)\) defined by
(4.2), we get
\[
V^0(x_0, t_0) - V^0(x_1, t_1) = \int_{t_0}^{t_1} [L(x, k(x, t), t) + \varepsilon(x, t)] dt \\
> \int_{t_0}^{t_1} L(x, k(x, t), t) dt,
\]
provided \((\phi(t; x_0, t_0), t) \in \mathcal{N}\) for all \(t \in [t_0, t_1]\). This contradicts the definition of \(V^0\), by the same argument as above, and establishes Theorem 4.2.

The essence of the arguments in this section is to replace the hamiltonian \(H\) by the hamiltonian \(H^0\) by eliminating \(u\) with the aid of the minimum operation (4.8). We shall call this the minimum principle.

5. Canonical Differential Equations; Pontryagin’s Theorem

At this stage, the solution of the optimal control problem is reduced to the problem of solving the Hamilton–Jacobi partial differential equation. Following Carathéodory’s program, one can go a step further and show that the optimal motions must be solutions of the characteristics of the Hamilton–Jacobi equation, which are a set of ordinary differential equations of order \(2n\). They are the Euler equations in canonical form—or simply the canonical equations—of the problem.

In this way, the determination of optimal motions reduces to the solution of the canonical equations. But in order to show that a given motion is really optimal, one must still construct—abstractly or explicitly—a solution of the Hamilton–Jacobi partial differential equation or, what is the same thing in view of Theorem 4.2, the function \(V^0(x, t)\). Moreover, the solution of the canonical equations does not provide the optimal control law for which, by (4.12), knowledge of \(V^0\) is essential.

Let \(G\) be a region in the phase space satisfying the hypotheses (i–v) of Theorem 4.2. Let \(\phi^0(t)\) be an optimal motion starting at some phase in \(G\) and eventually reaching a phase on \(S\) without leaving \(G\). We define
\[
\psi^0(t) = V^0_x(\phi^0(t), t).
\]  
(5.1)

Differentiating \(\psi^0(t)\) with respect to \(t\), we have
\[
\frac{d\psi^0(t)}{dt} = V^0_x(\phi^0(t), t) + V^0_{xx}(\phi^0(t), t) \cdot f(\phi^0(t), u^0(t), t). 
\]  
(5.2)
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Differentiating the Hamilton–Jacobi equation, regarded as an identity in \( V^0(x, t) \), with respect to \( x \) yields

\[
V^0_x(x, t) + H^0_p(x, V^0_p(x, t), t) + H^0_x(x, V^0_p(x, t), t) \cdot V^0_{xx}(x, t) = 0 \quad (5.3)
\]

throughout \( G \). Recalling the definition of \( H \), it follows that

\[
H^0_p = f + [L_u + p \cdot f_u]c_p. \quad (5.4)
\]

We wish to show that the bracketed term is zero. For technical reasons, \textit{we assume that the boundary of } \( U(t) \) \textit{is smooth in the space } \( \mathbb{R}^m \times T \).

Consider a point \((x_0, p_0, t)\) and the corresponding \( u_0^0 = c(x_0, p_0, t) \) \( \subseteq U(t) \) at which \( H \) assumes its unique absolute minimum. Recall that \( U(t) \) is closed. The following possibilities arise:

a. The point \( u_0^0 \) is interior to \( U(t) \). Then the first derivative of \( H \) with respect to \( u \) must vanish at \( u^0 \):

\[
L_u(x_0, u_0^0, t_0) + p_0 \cdot f_u(x_0, u_0^0, t_0) = 0. \quad (5.5)
\]

b. The point \( u_0^0 \) is on the boundary of \( U(t) \). There are now two subcases.

i. There is at least one point in every neighborhood of \((x_0, p_0, t_0)\) such that the corresponding \( u^0 \) is in the interior of \( U(t) \). Then (5.5) holds also at \((x_0, p_0, t_0)\) since \( L_u, f_u, c \) are continuous in all arguments.

ii. There is a neighborhood \( N \) of \((x_0, p_0, t_0)\) such that every \( u^0 \) corresponding to points in \( N \) lies on the boundary of \( U(t) \). In this case, throughout \( N \) we must have

\[
g^i(c(x, p, t), t) = 0, \quad i = 1, \ldots, q \leq m.
\]

Since the boundary of \( U(t) \) is smooth, we assume that the functions \( g^i \) are differentiable in both arguments and also that the determinant

\[
\left| \frac{\partial g^i(u, t)}{\partial u_j} \right|
\]

has rank \( q \) at the point \((u_0^0, t_0)\). Then the well-known Lagrange multiplier rule \([4, \text{p. 166}]\) implies that

\[
L_u(x_0, u_0^0, t_0) + p_0 \cdot f_u(x_0, u_0^0, t_0) + \sum_{i=1}^q \nu_i g^i_j(u_0^0, t_0) = 0, \quad (5.6)
\]

with \( \nu_i \neq 0 \). On the other hand, differentiating \( g^i(c(x, p, t), t) = 0 \) with respect to \( x \) and \( p \) shows that
\[ g^i_t(x, p, t) \cdot c_x(x, p, t) = 0, \]
\[ g^i_t(x, p, t) \cdot c_p(x, p, t) = 0, \quad i = 1, \ldots, q. \]

Combining the foregoing two equations, we have
\[ [L_u(x_0, u_0^0, t_0) + p_0 \cdot f_u(x_0, u_0^0, t_0)] \cdot c_x(x_0, p_0, t_0) = 0, \]
\[ [L_u(x_0, u_0^0, t_0) + p_0 \cdot f_u(x_0, u_0^C, t_0)] \cdot c_p(x_0, p_0, t_0) = 0. \] (5.7)

Hence we conclude that
\[ H_x(x, p, t, c(x, p, t)) = H^0_x(x, p, t), \]
\[ H_p(x, p, t, c(x, p, t)) = H^0_p(x, p, t) = f(x, p, t); \] (5.8)

these equations follow immediately from (5.7) in case (ii) and from (5.6) in the other cases.

In view of (5.8), utilizing also (5.2) and (5.3), we obtain the canonical equations:
\[ \frac{dx}{dt} = H^0_p(x, p, t) = H_p(x, p, t, c(x, p, t)), \]
\[ \frac{dp}{dt} = -H^0_x(x, p, t) = -H_x(x, p, t, c(x, p, t)), \] (5.9)

which could also be written as the identities
\[ \frac{d\phi^0(t)}{dt} = H_p(\phi^0(t), \psi^0(t), t, u^0(t)), \]
\[ \frac{d\psi^0(t)}{dt} = -H_x(\phi^0(t), \psi^0(t), t, u^0(t)). \] (5.10)

The last equations constitute a special case of the following result.

**Pontryagin’s Theorem** [8]. *If the motion \( \phi^0(t) \) is optimal with control \( u^0(t) \), then there must exist a function \( \psi^0(t) \) such that (5.10) is satisfied and in addition the relation
\[ H(\phi^0(t), \psi^0(t), t, u) \geq H(\phi^0(t), \psi^0(t), t, u^0(t)) \] (5.11)

must hold for all \( u \in U(t) \).*

Equation (5.11) is Pontryagin’s form of the minimum principle. It is proved [8], [11] by constructing a special first variation of the func-
tion \( u^0(t) \). (In Pontryagin's paper, \( H \) is defined, following the standard convention, as the negative of the quantity (4.6). For this reason, Pontryagin speaks of the "maximum" principle. We feel that the present choice of sign, which is motivated by the dynamic programming approach to the definition of \( V^0 \), is more natural.)

Note that Pontryagin's theorem is valid [11] without the strong smoothness assumptions concerning \( V^0 \). But in that case one cannot identify \( \psi^0(l) \) with \( V^0_x(\psi^0(l), l) \), and there remains a gap between the necessary condition represented by Pontryagin's form (5.10) of the Euler equations and the Hamilton–Jacobi–Carathéodory theory that we have sketched above.

Nevertheless, our theory can still be used for the effective solution of problems in which \( V^0(x, t) \) does not have continuous second derivatives throughout the phase space.

6. Solution of a Minimal-Time Problem

Consider the linear system (harmonic oscillator),

\[
\frac{dx_1}{dt} = x_2, \tag{6.1}
\]

\[
\frac{dx_2}{dt} = -x_1 + u_1(t),
\]

with

\[
|u_1(t)| \leq 1. \tag{6.2}
\]

Determine a control law taking the state of the system to the origin in the shortest possible time. See the minimal-time control problem in Section 2.

This celebrated problem seems to have been first mentioned by Doll [12] in 1943 in a U.S. Patent. The first rigorous solution of the problem appeared in 1952 in the doctoral dissertation of Bushaw [13]. Bushaw states that the problem does not fall within the framework of the classical calculus of variations, and he solves it by elementary but highly intricate direct geometric arguments.

The hamiltonian theory developed above can be applied quite simply to give a rigorous proof of Bushaw's theorem.

We rewrite Equations (6.1) in matrix form as

\[
\frac{dx}{dt} = Fx + Gu(t), \tag{6.3}
\]
where

\[ F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad g = G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

The minimum principle shows that the optimal control must satisfy the relation

\[ u^0(t) = -\text{sgn} \left[ g'\psi^0(t) \right], \quad (6.4) \]

where sgn is the scalar function of the scalar \( g'p \) that takes on the value 1 when \( g'p > 0 \), -1 when \( g'p < 0 \), and is undetermined when \( g'p = 0 \).

Since the problem is invariant under translation in time, we shall drop the arguments referring to the initial time, which can be taken as 0 for convenience. Instead of considering motions in the phase space \((x_1, x_2, t)\), we need to consider them only in the state space \((x_1, x_2)\).

First we determine all possible optimal motions passing through the origin. There are three of these: Either \( \phi^0(t) \) is identically zero, which is trivial, or \( \phi^0(t) \) is a solution of (6.1) with \( u^0(t) \) +1 or -1.

Let \( u^0(t) = 1 \). Then the motion of (6.1) passing through the origin is a circular arc \( \gamma^+ \) of radius 1 about the point \((1, 0)\) (see Fig. 1). To check whether this motion is really optimal, we must verify first of all that we have \( g'\psi^0(t) < 0 \) along the entire arc. Now (5.9b) in this case is

\[ \frac{dp}{dt} = -F'p, \quad (6.5) \]

which is independent of \( x \) and has the solution

\[ \psi^0(t) = \begin{bmatrix} \cos (t - t_0) & \sin (t - t_0) \\ -\sin (t - t_0) & \cos (t - t_0) \end{bmatrix} \psi^0(t_0). \quad (6.6) \]

It is clear that \( \psi^0(t) \) is periodic with period \( 2\pi \); therefore the largest interval over which we have \( g'\psi^0(t) < 0 \) is at most of length \( < \pi \). This is actually achieved by choosing \( \psi^0(0) = (\epsilon, 1) \), so that

\[ g'\psi^0(t) < 0 \quad \text{for all} \quad 0 \leq t < \pi - \epsilon. \quad (6.7) \]

Thus the necessary condition provided by the Euler equation (5.10) is satisfied along the arc \( \gamma^+ \) up to the state \((2, 0)\). (The remaining portion of the arc \( \gamma^+ \) is shown by a dashed curve in Fig. 1.)

Now we must establish a sufficiency condition; in other words, we must show that the arc \( \gamma^+ \) is indeed an optimal motion between the states \((0, 0)\) and \((2, 0)\).
Let us define the set $S_1$ by

$$S_1 = \{ x ; -0.1 < x_1 < 0.1, \ x_2 = 0 \},$$

as shown in Figure 1. We consider the problem of reaching $S_1$ in minimal time. Since $V_i^0 = 0$, the Hamilton–Jacobi equation for this problem is

$$V_2^0 \cdot (Fx + g) = -1,$$  \hspace{1cm} (6.8)

which has the solution

$$V_1^0(x_1, x_2) = \frac{\pi}{2} + \arctan \frac{1 - x_1}{x_2}.$$

The value of $V^0$ for $x_2 = 0$ is defined by its limit as $x_2 \to 0$ from negative values. Then $V^0 = 0$ on $S_1$, as required. Moreover, the region $G_1$, where $V^0$ is to satisfy (6.8), is taken as the semicircular band indicated by the crosshatching in Figure 1.

It follows from the necessary and sufficient conditions given in Section 4 that if we connect any state on $\gamma^+$ with $S_1$ by means of a motion of (6.1) that is distinct from $\gamma^+$ and remains entirely in $G_1$, then the value of $V$ in (2.3) necessarily is greater than $V^0$.

But if it is not possible to reach $S_1$ from $\gamma^+$ faster than by proceeding along $\gamma^+$ itself, the same is true a fortiori as concerns reaching the state $(0, 0)$ on $S_1$. Hence we have proved:

The motion $\gamma^+$ is optimal relative to the region $G_1$.

The same construction establishes the local optimality of the motion $\gamma^-$ (see Fig. 1).
Now we let $S_2 = \gamma^+$, $\lambda(x) = V_2^0(x)$, and we consider the minimal-time problem relative to $S_2$. All optimal motions, denoted by $\delta^-$ in Figure 1, necessarily correspond to $u^0 = -1$. They are circular arcs of radius 1 about the point $(-1, 0)$. Applying the Euler equations the same way as before, we find that all optimal arcs $\delta^-$ must terminate on the semicircle of radius 1 centered at $(-3, 0)$, which is part of the curve $\Gamma$ in Figure 1. The arcs $\delta^-$ therefore fill up a region $G_2$ bounded by $\gamma^+$, $\gamma^-$, and the semicircle centered at $(-1, 0)$ that connects $(2, 0)$ with $(-4, 0)$. If we calculate the time needed to reach $\gamma^+$ starting from a point to $G_2$ and proceeding along $\delta^-$, we get a smooth function $V_2^0(x)$ satisfying the Hamilton–Jacobi equation (6.8). Details are left to the reader. This proves that all motions consisting of an arc of $\delta^-$ and an arc of $\delta^+$ are optimal.

The construction can be continued in a similar fashion until it covers any point in the plane. The optimal control law is

$$
 u_1^0(x) = k(x) = \begin{cases} 
+1 & \text{below the curve } \Gamma \text{ composed of semicircles of radius 1 and on } \gamma^+; \\
-1 & \text{above the curve } \Gamma \text{ and on } \gamma^-.
\end{cases}
$$

(6.9)

On $\Gamma - (\gamma^+ \cup \gamma^-)$, the value of $u^0$ is not determined by the minimum principle; it is easily verified that the choice of $u^0$ on $\Gamma - (\gamma^+ \cup \gamma^-)$ is immaterial as long as $|u_1| \leq 1$.

The control law (6.9) is Bushaw's theorem.

It should be noted that the function $V_0^0$, which is determined piecewise as $V_1^0$, $V_2^0$, etc., is not continuously differentiable at a point $P$ on $\gamma^+$. The limit of $V_0^0$ is infinite if we approach $P$ from below $\gamma^+$ along points that lie on the continuation of $\delta^-$; and the same limit is finite if we approach $P$ from above $\gamma^+$ along $\delta^-$. As a result, the Euler equations (5.9) do not have continuous solutions along optimal motions; the conjugate vector $p$ receives an "impulse" on passing through $\Gamma$. But the more general proof [11] of Pontryagin's theorem shows that relations (5.10) remain true, so that

$$
u^0(t) = -\text{sgn} \left[ g'\psi^0(t; p_0, t_0) \right],$$

(6.10)

where the initial condition $p_0$ for the adjoint equation (6.5) may be defined by

$$
p_0 = k(x_0),
$$
in which $k$ is the optimal control law (6.9). Then $\psi^0(t)$ vanishes on $\Gamma$, which shows that $\psi^0$ cannot be interpreted as $V_2^0$.

In using Pontryagin's theorem in the form just mentioned as a neces-
sary condition to determine all possible optimal motions, it is still necessary to carry out the explicit construction given above, for (6.10) can be interpreted as the optimal control only if an optimal motion is known to exist connecting \( x_0 \) with the origin. In this very special case one can prove sufficiency without the Hamilton-Jacobi theory by noting that for any \((x_0, t_0) \) there is exactly one \( u^0(t) \) satisfying (6.10) which transfers \( x_0 \) to 0.

7. General Solution of the Linear Optimal Regulator Problem

A class of problems that can be completely solved by the hamiltonian theory is represented by the functional\(^\dagger\)

\[
\left\| \phi_u(t_1; x_0, t_0) \right\|^2_A + \frac{1}{2} \int_{t_0}^{t_1} \left[ \left\| H(t)\phi_u(t; x_0, t_0) \right\|_{Q(t)}^2 + \left\| u(t) \right\|_{R(t)}^2 \right] dt, \quad (7.1)
\]

where the motions \( \phi_u \) are defined by the linear differential equation

\[
\frac{dx}{dt} = F(t)x + G(t)u(t); \quad (7.2)
\]

there are no constraints on \( u \).

This is a slight generalization of the regulator problem given in Section 2.

The matrices \( Q(t) \), \( R(t) \) are taken to be positive definite for all \( t \). This assumption on \( R \) implies that the equation

\[
2H(x, p, t, u) = \left\| H(t)x \right\|_{Q(t)}^2 + \left\| u \right\|_{R(t)}^2 + 2p \cdot [F(t)x + G(t)u(t)] \quad (7.3)
\]

has a unique absolute minimum for every \((x, p, t)\) at

\[
c(x, p, t) = -R^{-1}(t)G'(t)p,
\]

so that we have

\[
2H^0(x, p, t) = \left\| H(t)x \right\|_{Q(t)}^2 + 2p \cdot F(t)x - \left\| G'(t)p \right\|_{R^{-1}(t)}^2. \quad (7.4)
\]

The Hamilton-Jacobi equation corresponding to (7.4) has a unique solution, given any nonnegative definite \( A \) and any \( t_1 > t_0 \). To show this, we assume that (4.9) has a solution of the form

\[
2V^0(x, t) = \left\| x \right\|_{P(t)}^2, \quad (7.5)
\]

\( ^\dagger \) We use the notation \( \left\| x \right\|_A^2 \) for a quadratic form defined by a symmetric non-negative definite matrix \( A \).
which implies the linear control law

\[ k(x, t) = -R^{-1}(t)G'(t)P(t)x. \]  

(7.6)

It is easily checked that (4.9) with the Hamiltonian defined by (7.4) has a solution of the type (7.5) if and only if the symmetric matrix \( P(t) \) is a solution of the Riccati equation

\[ -\frac{dP}{dt} = F'(t)P + PF(t) - PG(t)R^{-1}(t)G'(t)P + H'(t)Q(t)H(t). \]  

(7.7)

Moreover, the boundary condition

\[ V^0(x_1, t_1) = \|x\|^2, \]

which is the concrete form of (4.4), implies that the solution of (7.7) must satisfy the initial condition

\[ P(t_i) = 2A. \]  

(7.8)

Since (7.7) is nonlinear, it is not clear at once that \( P(t) \) exists outside of a small neighborhood of \( t_i \). The integral (7.1) may nevertheless be bounded from above by the free motions of (7.2), that is, by setting \( u(t) = 0 \), which in view of (7.5) is equivalent to a bound on \( \|P(t)\| \).

Using the a priori bound so obtained in the standard existence theorem for differential equations shows that solutions of (7.7) exist for all \( t \leq t_i \). This conclusion is in general no longer valid if \( A \) has negative eigenvalues or if \( t > t_i \).

Once the existence of solutions of (7.7), and therefore of the Hamilton-Jacobi equation, is ensured, the solutions can be expressed \([2], [3], [7]\) with the aid of solutions of the canonical differential equations

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dp}{dt}
\end{bmatrix} =
\begin{bmatrix}
F(t) & -G(t)R^{-1}(t)G'(t) \\
-H'(t)Q(t)H(t) & -F'(t)
\end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}.
\]

(7.9)

Further difficulties arise, however, in studying the stability of (7.7) as well as the stability of the optimal motions defined by (7.6). Details of these problems may be found particularly in \([7]\).

8. General Solution of the Linear Optimal Servomechanism Problem

The problem considered in the previous section can be generalized in several ways. We consider here simultaneously two such generalizations.

First, we assume that the motions, in addition to control, are subject to "disturbances" represented by the term \( w(t) \) in the equation
\[
\frac{dx}{dt} = F(t)x + G(t)u(t) + w(t). \tag{8.1}
\]

Second, we assume that the functional to be minimized is
\[
\left\| \eta(t_1) - H(t_1)\phi_u(t_1) \right\|^2_A + \frac{1}{2} \int_{t_0}^{t_1} \left[ \left\| \eta(t) - H(t)\phi_u(t) \right\|^2_{Q(t)} + \left\| u(t) \right\|^2_{R(t)} \right] dt. \tag{8.2}
\]

We call the \( p \)-vector,
\[
y(t) = H(t)x(t), \tag{8.3}
\]
the \textit{output} of the system (8.1); by analogy, the vector function \( \eta(t) \) is the \textit{desired output}.

This setup is a slight generalization of the servomechanism problem of Section 2. A number of formal solutions have appeared in the engineering literature [14], [15]. The Hamiltonian theory provides a simple rigorous proof of the known formulas.

Proceeding exactly as in Section 7, we find that the Hamiltonian of the problem is
\[
2H^0(x, p, t) = \left\| \eta(t) - H(t)x \right\|^2_{Q(t)} + 2p \cdot [F(t)x + w(t)] - \| G'(t)p \|_{R^{-1}(t)}. \tag{8.4}
\]

To solve the corresponding Hamilton–Jacobi equation (4.9), we assume that
\[
2V^0(x, t) = \| x \|^2_{P(t)} - 2z(t) \cdot x + \nu(t). \tag{8.5}
\]

Substituting, we obtain the following result:

\textbf{Theorem.} The function \( V^0(x, t) \) given by (8.5) satisfies the Hamilton–Jacobi equation defined by (8.4), with \( V^0(x, t) = \| \eta(t_1) - H(t_1)x \|^2_A \), if and only if

a. the matrix \( P(t) \) is the solution of the Riccati equation (7.7) with \( P(t_1) = 2A \);

b. the vector \( z(t) \) is the solution of
\[
\frac{dz}{dt} = -\left[ F(t) - G(t)R^{-1}(t)G(t)P(t) \right]z + P(t)w(t) - H'(t)Q(t)\eta(t), \tag{8.6}
\]

with
\[
H'(t_1)A \eta(t_1) = z(t_1); \tag{8.7}
\]
c. the scalar $v(t)$ is the solution of

$$\frac{dv}{dt} = \left(||\eta(t)||_{\mathcal{Q}(t)}^2 - ||G'(t)z(t)||_{R^{-1}(t)}^2\right) - 2z(t) \cdot w(t), \quad (8.8)$$

with

$$v(t_1) = 2||\eta(t_1)||_A^2.$$ 

The control law is linear, for it is given by

$$u^0(t) = -R^{-1}(t)G'(t)p(t) = R^{-1}(t)G'(t)[z(t) - P(t)z(t)]. \quad (8.9)$$

This law, however, is unrealizable, because it involves $z(t)$ which, according to (8.6) and (8.7), must be computed backward in time and requires the knowledge of $\eta(t)$ and $w(t)$ in the interval $[t_0, t_1]$ and this usually is not known at the time $t_0$ in practical applications.

It should be noted that the differential equation for $z(t)$, minus the forcing terms, is the adjoint of the differential equation of optimal motions given in Section 7.

References