This replaces pages 56 and 57:

**Theorem 1** Let $\Sigma$ be a (time-invariant, continuous-time) system, and pick any interval $[\sigma, \tau]$. Consider the map that gives the final state given an initial state and control,

$$\alpha : \mathcal{D}_{\sigma, \tau} \to \mathcal{X} : (x, \omega) \mapsto \xi(\tau) ,$$

as well as the mapping into the entire path

$$\psi : \mathcal{D}_{\sigma, \tau} \to \mathcal{C}_n^0 : (x, \omega) \mapsto \xi ,$$

where $\xi(t) = \phi(t, \sigma, x, \omega)(\sigma, t)$). If the system is of class $C^1$ and $(\xi, \omega)$ is any trajectory, and if

$$\lambda_0 \in \mathbb{R}^n \quad \text{and} \quad \mu \in \mathcal{C}_m^\infty ,$$

consider also the solution $\lambda : [\sigma, \tau] \to \mathbb{R}^n$ of the variational equation

$$\dot{\lambda}(t) = f_x(\xi(t), \omega(t))\lambda(t) + f_u(\xi(t), \omega(t))\mu(t) , \quad (2.35)$$

with initial condition $\lambda(\sigma) = \lambda_0$. The following conclusions then hold:

1. The set $\mathcal{D}_{\sigma, \tau}$ is an open subset of $\mathcal{X} \times \mathcal{L}_m^\infty$, and both $\psi$ and $\alpha$ are continuous.

2. Take any $x$ and any $\omega$ admissible for $x$, and denote $\xi := \psi(x, \omega)$. Let $\{\omega_j\}_{j=1}^\infty$ be an equibounded sequence of controls (that is, there is some fixed compact $K \subseteq \mathcal{U}$ such that $\omega_j(t) \in K$ for all $j$ and almost all $t \in [\sigma, \tau]$) and

$$\lim_{j \to \infty} x_j = x .$$

If either one of the following conditions hold:

(i) $\omega_j \to \omega$ as $j \to \infty$ pointwise almost everywhere, or

(ii) $\omega_j \to \omega$ as $j \to \infty$ weakly and $\Sigma$ is affine in controls,

then $\xi_j := \psi(x_j, \omega_j)$ is defined for all large $j$ and

$$\lim_{j \to \infty} \|\xi_j - \xi\|_\infty = 0 .$$

3. If $\Sigma$ is of class $C^1$, then $\alpha$ is of class $C^1$ on $\mathcal{X} \times \mathcal{L}_m^\infty$, and in that case

$$\alpha_*[x, \omega] (\lambda_0, \mu) = \lambda(\tau)$$

when $\lambda$ is as in (2.35). That is, $\alpha_*[x, \omega]$ is the same as the map $\alpha_*$ corresponding to the linearization $\Sigma_*[\xi, \omega]$. 


In particular, for systems of class \( \mathcal{C}^1 \),
\[
\alpha(x, \cdot) : D_{\sigma, \tau, x} \to \mathcal{X}
\]
has full rank at \( \omega \) if and only if the linear map
\[
\mathcal{L}_\infty \to \mathcal{X} : \mu \mapsto \int_\sigma^\tau \Phi(\tau, s)B(s)\mu(s)\,ds
\]
(the map \( \alpha(0, \cdot) \) for the linearization \( \Sigma_*[\Gamma] \) along \( \Gamma = (\xi, \omega) \), seen as a time-varying linear system) is onto.

**Proof.** First note that conclusion 1 follows from statement 2(i); this is because uniform convergence \( \omega_j \to \omega \) in \( L^\infty_{\mathcal{C}}(\sigma, \tau) \) implies in particular pointwise convergence, as well as — cf. Remark C.1.3 in the Appendix on ODE’s — equiboundedness of the sequence \( \{\omega_j\} \). We will show how the continuity statements 2(ii) and 2(ii) are both easy consequences of Theorem 37 found in that Appendix. Assume given any \((x, \omega) \in D_{\sigma, \tau}\) and any sequence \( \omega_j \to \omega \) converging in either of the two senses, and let \( \xi = \psi(x, \omega) \). Pick any \( \varepsilon > 0 \). We wish to show that \( \|\xi_j - \xi\|_\infty < \varepsilon \) for all large \( j \).

We start by choosing an open subset \( \mathcal{X}_0 \) of \( \mathcal{X} \) whose closure is a compact subset of \( \mathcal{X} \) and such that \( \xi(t) \in \mathcal{X}_0 \) for all \( t \in T := [\sigma, \tau] \). Let \( K \subseteq \mathcal{U} \) be a compact set so that \( \omega_j(t) \in K \) and \( \omega(t) \in K \) for almost all \( t \in T \). Introduce the functions \( \mathcal{T} \times \mathcal{X} \to \mathbb{R}^n \) given by
\[
h_j(t, z) := f(z, \omega_j(t)) - f(z, \omega(t))
\]
for each \( j = 1, 2, \ldots, \), and
\[
\bar{f}(t, z) := f(z, \omega(t)).
\]
Note that all these functions satisfy the hypotheses of the existence Theorem 36 (namely (H1), (H2), and the local Lipschitz and integrability properties) found in the Appendix on ODE’s. For each \( j \) let \( g_j := f + h_j \), so \( g_j(t, z) = f(z, \omega_j(t)) \). Since the Jacobian \( \|f_z\| \) is bounded on the compact set \( \text{clos} \mathcal{X}_0 \times K \), there is a constant \( a > 0 \) so that
\[
\|g_j(t, x_1) - g_j(t, x_2)\| \leq a \|x_1 - x_2\|
\]
for all \( t \in T \), all \( j \), and all \( x_1, x_2 \in \mathcal{X}_0 \). Let
\[
H_j(t) := \int_\sigma^t h_j(s, \xi(s))\,ds , \quad t \in T
\]
and \( H_j := \sup_{t \in T} \|H_j(t)\| \).

Assume for now that we have already proved the following fact:
\[
\lim_{j \to \infty} H_j = 0 \quad (2.36)
\]
(which amounts to saying that the functions $H_j$ converge uniformly to zero). We wish to apply Theorem 37, with “$A$” there being $\mathcal{A}_0$, “$f$” being $f$, $\alpha(t) \equiv \alpha$, and $z^0$ and $h$ equal to $x_j$ and $h_j$ respectively, for all large enough $j$. Take any $0 < D \leq \varepsilon$ so that
\[
\{z \mid \|z - \xi(t)\| \leq D \text{ for some } t \in [\sigma, \tau]\}
\]
is included in $\mathcal{A}_0$, and choose an integer $j_0$ so that, for each $j \geq j_0$, $\|x - x_j\|$ and $H_j$ are both less than
\[
\frac{\varepsilon}{2} e^{-a(\tau-\sigma)} \leq \frac{D}{2} e^{-a(\tau-\sigma)}.
\]
For any such $j$, using $z^0 = x_j$ and $h = h_j$ in Theorem 37 implies that
\[
\|\xi - \xi_j\|_\infty \leq (\|x - x_j\| + H_j) \leq \varepsilon
\]
as desired.

We now show (2.36). Suppose first that $\omega_j(t) \to \omega(t)$ a.e. as $j \to \infty$. Then $h_j(t, \xi(t)) \to 0$ a.e. as well. Moreover, letting $c$ be an upper bound on the values of $\|f\|$ on the compact set $\text{clos} \mathcal{A}_0 \times K$,
\[
\|h_j(t, \xi(t))\| \leq 2c
\]
for all $t \in \mathcal{I}$. The Lebesgue Dominated Convergence Theorem ensures that
\[
\int_\sigma^t \|h_j(s, \xi(s))\| \, ds \to 0,
\]
from which (2.36) follows.

Suppose now that $\Sigma$ is affine in controls and convergence is in the weak sense. We may write
\[
h_j(s, \xi(s)) = G(\xi(s)) [\omega_j(s) - \omega(s)],
\]
where $G(z)$ is the matrix whose columns are $g_1(z), \ldots, g_m(z)$. Thus the $i$th coordinate of $h_j(s, \xi(s))$, $i = 1, \ldots, n$, has the inner product form
\[
\varphi_i(s)' [w_j(s) - \omega(s)]
\]
where $\varphi_i(s)$, the transpose of the $i$th row of $G(\xi(s))$, is a continuous and hence integrable function of $s$. By the weak convergence assumption,
\[
\int_\sigma^t \varphi_i(s)' [w_j(s) - \omega(s)] \, ds \to 0 \text{ as } j \to \infty
\]
for each fixed $t \in \mathcal{I}$ and each $i = 1, \ldots, n$. Moreover, the functions $H_j$ form an equicontinuous family, because $\|H_j(t_1) - H_j(t_2)\| \leq 2c|t_1 - t_2|$ (recall that the norm of $h_j$ is upper bounded by $2c$). Pointwise convergence of an
equicontinuous function implies uniform convergence, so also in this case we have proved (2.36).

We next prove differentiability. (The proof is organized in such a manner as to make a generalization in Section 2.9 very simple.) Let \((x, \omega)\) be given, and \(\xi := \psi(x, \omega)\). Note that now \(X = \mathbb{R}^n\) and \(U = \mathbb{R}^m\). We first multiply \(f\) by a smooth function \(\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\) which is equal to one in a neighborhood of the set of values \((\xi(t), \omega(t))\) and has compact support. Since differentiability is a local property, it is sufficient to prove the result for the new \(f\) obtained in this manner. Thus from now on we assume that \(f\) is bounded and globally Lipschitz:

\[
|f(x, u) - f(z, v)| \leq M (|x - z| + |u - v|)
\]

for all \(x, z\) in \(\mathbb{R}^n\) and all \(u, v\) in \(\mathbb{R}^m\). In particular, solutions are globally defined.

(Proof continues in same manner as before, save for minor typos.) \(\blacksquare\)
This is to be added at the end of section C.1:

Remark C.1.3 Assume that $\mathcal{I}$ is an interval and $\mathcal{U}$ is a metric space. Suppose that we have a convergent sequence $\omega_i \to \omega$ in $\mathcal{L}^\infty_{u}(\mathcal{I})$. We prove here that the set of these controls is equibounded, that is, there is some compact subset $\bar{\mathcal{K}}$ of $\mathcal{U}$ so that, for almost all $t \in \mathcal{I}$, $\omega(t) \in \bar{\mathcal{K}}$ and $\omega_i(t) \in \bar{\mathcal{K}}$ for all $i$. By definition of $\mathcal{L}^\infty_{u}(\mathcal{I})$ there are compact subsets $K$ and $K_i$, $i \geq 1$, so that, except for $t \in \mathcal{I}$ in a set of zero measure, $\omega(t) \in K$ and $\omega_i(t) \in K_i$ for all $i$. For each $\varepsilon > 0$, let $B_\varepsilon$ be the set of points at distance at most $\varepsilon$ from $K$. Since $\omega_i \to \omega$ in uniform norm, for almost all $t$ we have that $\sup_{i \in \mathcal{I}} d(\omega_i(t), \omega(t)) < \varepsilon_i$, where $\{\varepsilon_i\}$ is a sequence of real numbers converging to zero, and so $\omega_i(t) \in B_{\varepsilon_i}$ for all $i$. (Observe that these sets need not be compact unless, for example, one assumes the space $\mathcal{U}$ to be locally compact.) Replacing $K_i$ by its intersection with $B_{\varepsilon_i}$, we may take each $K_i$ as contained in $B_{\varepsilon_i}$. It will be enough to show that the set $\bar{\mathcal{K}}$ defined as the union of $K$ and of all the sets $K_i$ is also compact. We pick a sequence $\{u_i\}$ of elements of $\bar{\mathcal{K}}$ and show that it admits a convergent subsequence.

If this sequence is contained in a finite union $A = K_1 \cup K_2 \cup \ldots \cup K_r$, then it has a convergent subsequence, since $A$ is compact, and we are done. So assume that the sequence is not contained in any finite union of this form. Then, there are subsequences $u_{j_1}$ and $K_{k_1}$ so that $u_{j_1} \in K_{k_1}$ for all $\ell$. (Such a subsequence can be obtained by induction: if we already have $j_1, \ldots, j_\ell$ and $k_1, \ldots, k_\ell$, let $A = K_1 \cup K_2 \cup \ldots \cup K_{k_\ell}$ and let $B$ be a union of finitely many $K_i$’s which contains $u_1, \ldots, u_{j_\ell}$; as the entire sequence cannot be contained in the union of $A$ and $B$, there are $j > j_\ell$ and $k > k_\ell$ so that $u_j \in K_k$, so we may take $j_{\ell+1} := j$ and $k_{\ell+1} := k$.) We renumber and suppose from now on that $u_i \in K_i$ for each $i$. Pick for each $i = 1, 2, \ldots$ some $y_i \in K_i$ such that $d(u_i, y_i) \leq \varepsilon_i$. Since $K$ is compact, there is a subsequence of $y_1, y_2, \ldots$ which converges to some element $y \in K$, and renumbering again, we assume without loss of generality that $y_i \to y$ as $i \to \infty$. Thus $u_i \to y$ as well, so the original sequence has a convergent subsequence, establishing compactness.

This replaces the previous section with same number:

C.4 Continuous Dependence

We now prove that solutions depend continuously on the form of the right-hand side of the equation.

Theorem 2 Let $\alpha : \mathcal{I} \to \mathbb{R}_+$ be an integrable function, where $\mathcal{I} = [\sigma, \tau]$ is a bounded closed interval in $\mathbb{R}$, $\mathcal{X}$ an open subset of $\mathbb{R}^n$, and $D$ a positive real number. Suppose that $f$ and $h$ are two mappings $\mathcal{I} \times \mathcal{X} \to \mathbb{R}^n$ which satisfy the hypotheses of Theorem 36 (namely (H1), (H2), and the local
Lipschitz and integrability properties), and that \( \dot{\xi} : \mathcal{I} \to \mathcal{X} \) is a solution of
\[
\dot{\xi}(t) = f(t, \xi(t))
\]
such that the \( D \)-neighborhood of its range:
\[
K = \{ x \mid \| x - \xi(t) \| \leq D \text{ for some } t \in [\sigma, \tau] \}
\]
is included in \( \mathcal{X} \). Let
\[
H(t) := \int_\sigma^t h(s, \xi(s)) \, ds, \quad t \in \mathcal{I}
\]
and \( \mathcal{H} := \sup_{t \in \mathcal{I}} \| H(t) \| \). Assume that
\[
\max \{ H, \| \xi(\sigma) - z^0 \| \} \leq \frac{D}{2} e^{-\int_\sigma^\tau a(s) \, ds} \tag{C.16}
\]
and, with \( g := f + h \),
\[
\| g(t, x) - g(t, z) \| \leq a(t) \| x - z \| \text{ for all } x, z \in \mathcal{X} \text{ and } t \in \mathcal{I}. \tag{C.17}
\]
Then, the solution \( \zeta \) of the perturbed equation
\[
\dot{\zeta} = g(t, \zeta) = f(t, \zeta) + h(t, \zeta) , \quad \zeta(\sigma) = z^0 \tag{C.18}
\]
exists on the entire interval \( [\sigma, \tau] \), and is uniformly close to the original solution in the following sense:
\[
\| \xi - \zeta \|_\infty < \left( \| \xi(\sigma) - z^0 \| + \mathcal{H} \right) e^{\int_\sigma^\tau a(s) \, ds}. \tag{C.19}
\]

**Proof.** Let \( \zeta : J \to \mathbb{R}^n \) be the maximal solution of the initial value problem (C.18), \( J \subseteq \mathcal{I} \). This solution exists because \( f \) and \( h \), and therefore \( g \), satisfy the hypotheses of the existence theorem. For any \( t \in J \) we have:
\[
\xi(t) - \zeta(t) = \xi(\sigma) - z^0 + \int_\sigma^t [g(s, \xi(s)) - g(s, \zeta(s))] \, ds - \int_\sigma^t h(s, \xi(s)) \, ds.
\]
Therefore, for each \( t \in J \):
\[
\| \xi(t) - \zeta(t) \| \leq \| \xi(\sigma) - z^0 \| + \| H(t) \| + \int_\sigma^t a(s) \| \xi(s) - \zeta(s) \| \, ds
\]
and hence from Gronwall’s inequality we conclude:
\[
\| \xi(t) - \zeta(t) \| \leq \left( \| \xi(\sigma) - z^0 \| + \mathcal{H} \right) e^{\int_\sigma^\tau a(s) \, ds} \tag{C.20}
\]
In particular, this implies that \( \| \xi(t) - \zeta(t) \| \leq D \), and therefore \( \zeta(t) \in K \). Thus the maximal solution \( \zeta \) is included in a compact subset of \( \mathcal{X} \), which by Proposition C.3.6 implies that \( J = \mathcal{I} \), and this means that Equation (C.20) holds for all \( t \in \mathcal{I} \), which is what Equation (C.19) asserts. \[ \square \]