Well-defined steady-state response does not imply CICS

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Abstract

Systems for which each constant input gives rise to a unique globally attracting equilibrium are considered. A counterexample is provided to show that inputs which are only asymptotically constant may not result in states converging to equilibria (failure of the converging-input converging-state, or “CICS” property).

Keywords: Step response; Global attractivity; Homoclinic orbits; Asymptotically autonomous systems

1. Introduction

Consider a controlled finite dimensional system

\[ \dot{x}(t) = f(x(t), u(t)), \]  

(1)

under suitable regularity assumptions, and assume that the following property holds: for each constant input \( u \equiv a \), there is a unique steady-state \( x_a \) (that is, \( f(x, a) = 0 \)) has the unique solution \( x = x_a \), and every solution of the system \( \dot{x} = f(x, a) \) converges to this state \( x_a \). In the terminology of [1–3], we say that the system has a “characteristic” or a monostable steady-state step response.

Given that (1) admits a characteristic, it is natural to ask if the following converging-input converging-state (CICS) property must then also hold: for every convergent input \( u(\cdot) \) (that is to say, \( u(t) \to a \) as \( t \to \infty \), for some value \( a \)), every bounded solution of \( \dot{x} = f(x, u) \) converges to \( x_a \).

Such a property is especially interesting when studying cascades of systems, in which the input \( u \) to the system being studied is itself the output of another system. That is, there is another system \( \dot{z} = g(z, v) \), \( u = h(z) \), and \( v \) is an external input to the cascade. In that context, one would like to know whether each of the \( f \) and \( g \) systems having a characteristic implies that the cascade also does. Suppose that \( v \equiv a \). If the \( g \) system has a monostable response, its state converges to some value: \( z(t) \to z_a \), so that also, assuming continuity of the read-out map \( h, u(t) \to b := h(z_a) \). If the CICS property holds for the \( f \) subsystem (and assuming that its trajectories are bounded), then \( x(t) \to x_b \), and therefore the complete state \( (z(t), x(t)) \) converges to \( (z_a, x_b) \), establishing that the cascade also admits a characteristic.

These questions have a long history in control as well as in dynamical systems theory, see for example the early work of Markus [6], and are closely related to the topic of “asymptotically autonomous” systems, see for example [4] Appendix A (by Z. Artstein). The latter are time-varying systems \( \dot{x} = F(x, t) \) for which \( F(x, t) \to F_0(x) \) as \( t \to \infty \), for some time-invariant vector field \( F_0 \), where the convergence is assumed to hold in an appropriate technical sense. Clearly, one may view \( f(x, u(t)) \), for any fixed given input \( u(\cdot) \), as a time-varying vector field \( F(x, t) \), and, if \( u(t) \to a \) as \( t \to \infty \), one may define \( F_0(x) := f(x, a) \); in this manner, “\( u(t) \to a \)” translates into “\( F(x, t) \to F_0(x) \),” and the questions addressed here amount to relating the behaviour of solutions of \( \dot{x} = F(x, t) \) to that of solutions of the limit system \( \dot{x} = F_0(x) \). For other related work, see for example [7–9,11,12,5].

There are several known sufficient conditions that guarantee the CICS property for systems which admit characteristics. One such condition is stability of the equilibria \( x_a \). That is, not only...
do trajectories of $\dot{x} = f(x, a)$ approach $x_a$ as $t \to \infty$, but the “small excursion” Lyapunov stability condition holds as well: for each neighbourhood $U$ of $x_a$, there is another neighbourhood $V$ such that solutions starting in $V$ do not exit $U$ (later in this paper we discuss a weaker condition, which is implied by but does not imply the stability condition). The conjunction of stability and global attractivity of $x_a$ is, of course, equivalent to global asymptotic stability of $x_a$ under which condition the CICS property is a particular consequence of Theorem 2 in [6]. A different condition ensuring the CICS property is that of monotonicity: the conclusions hold provided that the system is monotone as an input/output system in the sense of [2]; the paper [2] made stability into part of the definition of characteristic, but [3] showed one need not assume stability in order to conclude the CICS property.

In view of these different sufficient conditions, it is natural to ask if it is always true that the CICS property holds for systems with characteristics. The main goal of this note is to provide a simple criterion which guarantees the CICS property.
then only on the annulus \( \{(r, \theta) | a/2 < r < a, 0 \leq \theta < 2\pi\} \). Since \( r(\cdot) \) is strictly decreasing (with limit 1) along all solutions exterior to the closed unit disc, for \( u = a > 0 \) we have \( r(t) - 1 \leq a/2 \) for all \( t \) sufficiently large and so the qualitative behaviour of the above figure is ultimately exhibited. We may now conclude that, for every constant input, \((1, 0)\) is a globally attractive equilibrium.

Fix \( r > 1 \) arbitrarily and write \( c := 2(r - 1) \). The following construction yields a bounded input \( u \), with \( u(t) \to 0 \) as \( t \to \infty \) (and \( u \in L^p([0, \infty)) \) for all \( q > p \)), such that, with initial data \( (r(0), \theta(0)) = (\theta^0, \theta^0) \), \( 0 \leq \theta^0 < 2\pi \), the corresponding solution of \((3)\) has \( S^1 \) as its \( \omega \)-limit set.

The initial-value problem \( \dot{r} = -(r - 1)^{p+1}, \ r(0) = r^0 \), has unique solution
\[
 t \mapsto r(t) = 1 + \frac{r^0 - 1}{(1 + p(r^0 - 1)^{p})^{1/p}}
\]
(strictly decreasing and converging to 1 as \( t \to \infty \)). Observe that (by \((4)\)), if \( r(t) = 1 + (c/2^n) \), then \( r(t + \tau_n) = 1 + (c/2^n) \), where
\[
 \tau_n = \frac{2^n p (2^n - 1)}{p c^p} > \frac{2^n p}{c^p} \quad \forall n \in \mathbb{N}.
\]
Define the sequence \((\tau_n)\) by
\[
 r(\tau_n) = 1 + (c/2^n) \quad \forall n \in \mathbb{N}.
\]
Then, \( \tau_1 = 0 \) and, by the previous observation,
\[
 \tau_{n+1} - \tau_n = \tau_n \quad \forall n \in \mathbb{N}.
\]
Define the function \( n : [0, \infty) \to (0, \infty) \) by the property
\[
 n \in \mathbb{N}, \quad t \in [\tau_n, \tau_{n+1}) \implies u(t) = \frac{c}{2^{n+1}}.
\]
Let \((r, \theta)\) be the solution on \([0, \infty)\) of \((3)\) with input \( u \) and initial data \( (r(0), \theta(0)) = (\theta^0, \theta^0) \), \( 0 \leq \theta^0 < 2\pi \). For all \( n \in \mathbb{N} \), \( (r(t) - 1)/u(t) \in \{|1, 2|\} \) for all \( t \in [\tau_n, \tau_{n+1}) \) and so
\[
 g(r(t), u(t)) = u(t) = \frac{c^p}{2^{n+1} p} \quad \forall t \in [\tau_n, \tau_{n+1}).
\]
Since
\[
 \dot{\theta}(t) = 4 \sin^2(\theta(t)/2) + g(r(t), u(t)) \geq g(r(t), u(t)) \quad \forall t \geq 0,
\]
it follows that
\[
 \theta(\tau_{n+1}) - \theta(\tau_n) \geq \frac{\tau_n c^p}{2^{n+1} p} > \frac{2^{n+1} p}{c^p} = \frac{1}{2} \quad \forall n \in \mathbb{N}.
\]
Therefore, the increasing function \( \theta \) is unbounded. Since \( r(t) \to 1 \) as \( t \to \infty \), we may infer that \( S^1 \) is the \( \omega \)-limit set of the solution \((r, \theta)\).

3. A sufficient condition

In this section, we provide a sufficient condition—which, obviously, is violated in our counterexample—which guarantees the CICS property when characteristics exist. Our condition is not original; indeed, it may be viewed as a consequence of a more general result given by Thieme, cf. [11]. Nonetheless, it seems appropriate to provide it here, since it is much more elementary, and easier to state and check, than the general result in [11], and we can provide a simple self-contained proof.

We assume given a system \((1)\), whose states \( x(t) \) evolve on Euclidean space \( \mathbb{R}^n \), for some \( n \in \mathbb{N} \), and whose input takes values \( u(t) \) on some (locally compact) metric space \( U \). We assume that \( f : \mathbb{R}^n \times U \to \mathbb{R}^n \) is continuous and is locally Lipschitz on \( x \) uniformly on compact subsets of \( U \). By an \( \text{input} \) we mean a locally essentially bounded Lebesgue-measurable function \( u : [0, \infty) \to U \). Under these conditions, one knows (see for instance [10, Chapter 2]) that, for any initial state \( x^0 \in \mathbb{R}^n \) and input \( u \), there exists a unique solution of \((1)\) with \( x(0) = x^0 \), maximally defined on some interval \([0, \sigma_{x^0, u}]\), which we denote by \( x(\cdot) = \phi(\cdot, x^0, u) \); we write
\[
 S(x^0) := \{ \phi(t, x^0, u) | t \in [0, \sigma_{x^0, u}] \}
\]
for the forward orbit of \( x^0 \) under the input \( u \).

Fix a \( u : [0, \infty) \to U \) which converges: \( u(t) \to a \) as \( t \to \infty \), where \( a \in U \) so that \( f(x_a, a) = 0 \) for some \( x_a \in \mathbb{R}^n \). For notational simplicity and without loss of generality, we may assume \( x_a = 0 \). We will use the notation \(|\mu|\) to indicate the distance dist \((\mu, a)\) from any \( \mu \in U \) to \( a \), and also for norm in \( \mathbb{R}^n \), and write \( u_a \) for the constant input \( u_a(t) \equiv a \). We write \( g(x) = f(x, a) \) and let \( \phi_a \) denote the flow generated by \( \dot{x} = g(x) \) (and so \( \phi_a(x^0) = x^0 \)) is the maximal solution, forwards and backwards in time, of \( \dot{x} = g(x) = f(x, a) \) with initial state \( x^0 \), and, for \( t \geq 0 \), \( \phi_a(t, x^0) = \phi(t, x^0, u_a) \).

For any given \( x^0 \in \mathbb{R}^n \) for which \( S(x^0) \) is bounded (in which case, \( \sigma_{x^0, u_a} = +\infty \)), we consider the \( \omega \)-limit set of the trajectory of \( \dot{x} = f(x, u) \) starting from \( x^0 \):
\[
 \Omega(x^0) = \{ \xi | \phi(t, x^0, u) \to \xi \text{ for some sequence } t_j \to \infty \}.
\]
This is a nonempty compact set, and \( \phi(t, x^0, u) \to \Omega(x^0) \) as \( t \to \infty \), because of the boundedness assumption. The following useful fact is as in Markus [6].

Lemma 3.1. If \( S(x^0) \) is bounded, then the set \( \Omega(x^0) \) is invariant (both forward and backward in time) under the flow \( \phi_a \) of \( \dot{x} = g(x) \).

Proof. Pick any point \( \xi \in \Omega(x^0) \), and a sequence \( t_j \to \infty \) so that \( \phi(t_j, x^0, u) \to \xi \). Pick any \( T \in \mathbb{R} \) for which \( \xi := \phi_a(T, \xi) \) is defined. We will prove that \( \xi \in \Omega(x^0) \) (which will imply that \( \phi_a(T, \xi) \) is defined for all \( T \in \mathbb{R} \), because the maximal solution is defined globally once that it is known that it remains inside a compact set) and the invariance result follows. So, we need to show that there is a sequence \( t_j \to \infty \) such that \( \phi(T, x^0, u) \to \xi \), that is, for any \( \varepsilon > 0 \), we need to find a \( J \) such that \( |\phi(T, x^0, u) - \xi| < \varepsilon \) for all \( j > J \). Let \( \varepsilon > 0 \) be arbitrary. By continuity of \( \phi(T, \cdot, \cdot) \) with respect to states and uniform norms on inputs (see e.g. [10, Theorem 1]), and since \( \zeta := \phi(T, \xi, u_a) \), we know that there is some \( \delta > 0 \) so that \( |\phi(T, x, v) - \zeta| < \varepsilon \) whenever \( |x - \zeta| < \delta \) and \( t \to v(t) \in U \) is so that \( |v(t)| < \delta \) for all \( t \). Pick \( J \in \mathbb{R} \) so that, for all \( j > J \), \( T_j := t_j + T > 0, |\phi(t_j, x^0, u) - \zeta| < \delta \) and \( |u(t_j + t_j)| < \delta \) for all
\( t \geq 0 \). Noting that \( \phi(t_j, x^0, u) = \phi(T, x, v) \) with \( x = \phi(t_j, x^0, u) \) and \( v(t) = u(t + t_j) \), it follows that \( |\phi(T, x^0, u) - \zeta| < \varepsilon \) for all \( j > J \). \( \Box \)

Now, convergence \( \phi(t, x^0, u) \to 0 \) as \( t \to \infty \) is the same as asking that \( \Omega(x^0) = \{ 0 \} \) which, in conjunction with Lemma 3.1, yields the following condition that guarantees the CICS property.

**Corollary 3.2.** Assume that \( \dot{x} = g(x) \) admits no compact invariant set different from \( \{ 0 \} \). Then, for each \( x^0 \in \mathbb{R}^n \) for which \( S(x^0) \) is bounded, \( \phi(t, x^0, u) \to 0 \) as \( t \to \infty \).

There is an equivalent way to state this sufficient condition, which can be interpreted as ruling out homoclinic orbits in a generalised sense. Recall that the \( \alpha \)-limit set \( A(x^0) \) of the solution \( \phi_x(t, x^0) \) of \( \dot{x} = g(x), x(0) = x^0 \), consists of all points \( \zeta \) such that \( \phi_x(t_j, x^0) \to \zeta \) for some sequence \( t_j \to -\infty \), and, provided that the solution is bounded, this is a compact invariant set to which \( \phi_x(t, x^0) \) converges as \( t \to -\infty \).

**Proposition 3.3.** Assume that the system \( \dot{x} = g(x) \) has the property that, for each \( x^0 \in \mathbb{R}^n \), the solution \( t \mapsto \phi_x(t, x^0) \) is globally defined and converges to 0 as \( t \to \infty \). Then, the following two properties are equivalent:

- There is no nontrivial (different from \( \{ 0 \} \)) compact invariant set.
- There is no nontrivial bounded orbit for which \( 0 \) is an \( \alpha \)-limit point.

**Proof.** Assume that there is no nontrivial compact invariant set, and suppose that there exists an \( x^0 \neq 0 \) with bounded orbit for which \( 0 \in A(x^0) \). As \( A(x^0) \) is a compact invariant set, \( A(x^0) = \{ 0 \} \). Therefore, \( \phi_x(t, x^0) \to 0 \) as \( t \to -\infty \), and we also have that \( \phi_x(t, x^0) \to 0 \) as \( t \to +\infty \). Therefore, the set \( C \) which is made up of the orbit of \( x^0 \) together with the origin is a compact invariant set, and is nontrivial since \( x^0 \in C \). This gives a contradiction.

Conversely, assume that there is no nontrivial bounded orbit for which \( 0 \) is an \( \alpha \)-limit point, and suppose that there is a nontrivial compact invariant set \( C \). Pick any nonzero \( x^0 \in C \). The orbit of \( x^0 \) is a subset of \( C \), so it is bounded; thus \( A(x^0) \) is compact and invariant. Pick any \( \zeta \in A(x^0) \). Since \( \phi_x(t_j, \zeta) \to 0 \) as \( t \to \infty \) and \( A(x^0) \) is a closed invariant set, it follows that \( 0 \in A(x^0) \). So the orbit of \( x^0 \) is nontrivial and has 0 as an \( \alpha \)-limit point. This gives a contradiction. \( \Box \)

Next, we remark on how these conditions relate to the far deeper results in Thieme’s work [11] (see also [12,7]). Restricted to our context (Thieme deals with more complex attractors as well), Corollary 4.3 of [11] says that the statement “\( \phi(t, x^0, u) \to 0 \) as \( t \to \infty \) for each \( x^0 \in \mathbb{R}^n \) for which \( S(x^0) \) is bounded” may be concluded from the conjunction of the following two properties, for \( \dot{x} = g(x) \): (a) there are no nontrivial homoclinic orbits and (b) the set \( \{ 0 \} \) is an isolated compact invariant set. Since the closure of a homoclinic orbit is compact and invariant, (a) would follow from the stronger assumption that there are no nontrivial compact invariant subsets and (b) also follows from this same stronger assumption.

Finally, we re-interpret the above results in the specific context of systems with characteristics. Assume that \( f \) is so that (1) admits a characteristic. Then, for each constant input \( u \equiv a \in U \), there is a unique steady-state \( x_0 \) and every solution of \( \dot{x} = f(x, a) \) converges to \( x_0 \) as \( t \to \infty \). The final proposition is now an easy consequence of Corollary 3.2 and Proposition 3.3.

**Proposition 3.4.** The following properties are equivalent:

- for all \( a \in U \), \( \{ x_a \} \) is the only non empty compact set that is invariant under \( \dot{x} = f(x, a) \),
- for all \( a \in U \), \( x_a \) is not an \( \alpha \)-limit point of any bounded orbit of \( \dot{x} = f(x, a) \) other than the equilibrium orbit \( \{ x_a \} \),

and each implies that (1) has the CICS property.

Observe that, if \( x_a \) is a Lyapunov stable equilibrium of \( \dot{x} = f(x, a) \) for all \( a \in U \), then the second (and hence also the first) of the above properties holds. Therefore, the fact that stability of all equilibria \( x_a \) is sufficient to conclude the CICS property for systems with characteristics is a particular consequence of Proposition 3.4.

**References**


