Non-Smooth Stabilizers for Nonlinear Systems with Uncontrollable Unstable Linearization

Chunjie Qian and Wei Lin

Department of Electrical Engineering and Computer Science
Case Western Reserve University, Cleveland, Ohio 44106

Abstract

We prove that every chain of odd power integrators perturbed by a C^1 triangular vector field can be stabilized in the large via continuous state feedback, although it is not stabilizable, even locally, by any smooth state feedback. The proof is constructive and accomplished by developing a machinery—a continuous type of adding a power integrator—that enables one to explicitly design a C^0 globally stabilizing feedback law as well as a C^1 control Lyapunov function which is positive definite and proper.

1 Introduction

In this paper we consider a class of highly nonlinear systems in the lower-triangular form

\[
\begin{align*}
\dot{x}_1 &= x_2^p_1 + f_1(x_1) \\
\dot{x}_i &= x_2^p_{i-1} + f_{i-1}(x_1, \ldots, x_{i-1}) \\
\dot{x}_n &= u^{p_n} + f_n(x_1, \ldots, x_n),
\end{align*}
\]

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) and \( u \in \mathbb{R} \) are the system state and the control input, \( p_i, i = 1, \ldots, n \), are arbitrarily odd positive integers, and \( f_i : \mathbb{R} \to \mathbb{R}^i, i = 1, 2, \ldots, n \), are C^1 functions with \( f_i(0, \ldots, 0) = 0 \).

The main purpose of the paper is to address the questions: (i) when is there a continuous (i.e. C\(^0\)) state feedback control law that renders the trivial solution \( x = 0 \) of (1.1) globally strongly stable (GSS for short) in the sense of Kurzweil [4] (see the precise definition in Section 2)? (2) how to explicitly construct such a continuous, globally strongly stabilizing controller if there exists one?

Our interest in these two questions is motivated by a number of papers and books [1, 3, 12, 13, 18, 20], [5]-[16], which are devoted to the problem of local asymptotic stabilization via continuous or homogeneous state feedback, for lower-dimensional (two and three-dimensional) systems or lower-triangular systems whose first approximation has uncontrollable, unstable modes. The best known example, which has been extensively studied in the literature, is the planar system

\[
\begin{align*}
\dot{x}_1 &= x_2^3 + x_1 \\
\dot{x}_2 &= u.
\end{align*}
\]

This system has an uncontrollable mode whose eigenvalue is positive, and thus the system (1.2) cannot be stabilized, even locally, by any smooth (or C^\(r \geq 1\)) state feedback control laws [2].

While no smooth state feedback laws can locally asymptotically stabilize the system (1.2), it has been proved in [12, 13] that there exists, however, a locally Hölder continuous (non-Lipschitz) stabilizing controller for (1.2). An explicit algorithm was given in [12, 13], for the construction of Hölder continuous state feedback control laws that achieve local asymptotic stability, for a class of small-time locally controllable (STLC for short) affine systems in the plane, even if the Jacobian linearization has uncontrollable modes associated with eigenvalues whose real part is positive.

The results of [12, 13] (also see the survey paper [19]) suggest that continuous feedback design seems to be a natural strategy to overcome some topological obstruction such as the one illustrated by (1.2), which may occur in the case of smooth feedback stabilization (e.g. see [15]). This appealing idea, together with the powerful notions such as homogeneous approximation and homogeneity with respect to a family of dilations, [8, 10, 12, 13] has led to various exciting developments [1, 5, 6, 7, 8, 9, 10, 18] in the area of asymptotic stabilization of inherently nonlinear systems (e.g. (1.1) and STLC affine systems in \( \mathbb{R}^2 \)) having uncontrollable linearization, via continuous or non-differentiable state feedback.

Most of the stabilization results obtained in [1, 5, 6, 8, 9, 10, 12, 13] for lower-dimensional systems are local, due to the use of a homogeneous approximation. In the higher-dimensional case, the paper [7] studied the asymptotic stabilization of lower-triangular systems (1.1). It was shown that if \( p_i, 1 \leq i \leq n \), are odd positive integers, (1.1) is locally asymptotically stable by continuous state feedback. The result was proved by using Hermite's theorem on robust stability of homogeneous systems [9], and is basically an existence result. An important question on how to design a local continuous stabilizer remained, however, unknown.

The issue was addressed later in [3], where a formula was provided for the calculation of C^0 local controllers. Recently, the paper [22] investigated an interesting question on how to design a C^0 globally stabilizing controller for a class of homogeneous systems. In particular, it was proved in [22], among other things, that a subclass of nonlinear systems of the form (1.1) with \( f_i(x_1, \ldots, x_i) \) being a suitable linear function of the state \( (x_1, \ldots, x_i) \) are homogeneous with
respect to certain dilation, and can thus be stabilized in the large by \( C^0 \) homogeneous state feedback. The proof of the result in [22] relied crucially on a homogeneous-like Lyapunov function motivated by \[7\].

In a different direction, we considered in [15] the problem of global asymptotic stabilization via smooth state feedback, for a class of high-order nonlinear systems of the form (1.1). Using a design tool called adding a power integrator, we showed how to explicitly construct a globally stabilizing \( C^\infty \) state feedback control law for the system (1.1), under appropriate sufficient conditions which turn out to be somewhat necessary for smooth feedback stabilization [15].

In this work we focus our attention on the problem of global stabilization via continuous instead of smooth (e.g. [11, 15, 16]) state feedback. By combining the idea of the use of homogeneous-based Lyapunov functions [22] and the adding a power integrator technique [15], we develop a new, systematic design algorithm that simultaneously constructs a \( C^1 \) control Lyapunov function, which is positive definite and proper, and a \( C^0 \) state feedback control law that renders the trivial solution of the triangular system (1.1) globally strongly stable in the sense of Kurzweil [4]. This in turn leads to a rather surprising but important conclusion on global stabilization of nonlinear systems: every chain of odd power integrators perturbed by a \( C^1 \) lower-triangular vector field is globally strongly stabilizable by continuous state feedback.

The paper is organized as follows: In Section 2, we review a concept related to the general notions of stability and strong stability introduced by Kurzweil in the continuous framework [4]. We then introduce three technical lemmas that will be frequently used in the paper. The main result of the paper is given in Section 3, where a continuous state feedback control law is explicitly constructed, making the trivial solution of the lower-triangular system (1.1) globally strongly stable. Section 4 includes two simple yet challenging examples for which global stabilization, to the best of our knowledge, cannot be achieved by any existing continuous feedback control schemes but by the new strategy proposed in this work. Concluding remarks are drawn in Section 5.

2 Preliminaries

The classical Lyapunov stability theory (e.g. see [8, 14]) and the well-known concepts such as stability and asymptotic stability in the sense of Lyapunov can only be applied to a nonlinear differential equation whose solution from any initial condition is unique. However, a differential equation

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{2.1}
\]

with \( f : \mathbb{R}^n \to \mathbb{R}^n \) being a continuous, non-Lipschitz mapping and \( f(0) = 0 \), may have more than one solution starting from a given initial condition (e.g. \( \dot{x} = x^3/3 \) with \( x(0) = 0 \)), it is therefore necessary to introduce different (from Lyapunov) notions of stability and asymptotic stability in the continuous framework. In [4] Kurzweil introduced the new notions of stability for continuous systems (2.1) and established Lyapunov’s second theorem as well as the converse theorem of Lyapunov on stability, without requiring uniqueness of the trajectory of (2.1). In what follows, we recall Kurzweil’s definition on global strong stability which will be used in the next section.

Definition 2.1 (pp. 69, [4] or pp. 469 in [18]) The trivial solution \( x = 0 \) of (2.1) is globally strongly stable (GSS) if there are two functions \( B : (0, +\infty) \to (0, +\infty) \) and \( T : (0, +\infty) \times (0, +\infty) \to (0, +\infty) \) with \( B \) being increasing and \( \lim_{s \to 0} B(s) = 0 \), such that \( \forall \beta > 0 \) and \( \forall \varepsilon > 0 \), for every solution \( x(t) \) of (2.1) defined on \([0, t_1]\), \( 0 < t_1 \leq +\infty \) with \( \|x(0)\| \leq \beta \), there exists a solution \( z(t) \) of (2.1) defined on \([0, +\infty)\) satisfying the following:

(i) \( z(t) = x(t), \quad t \in [0, t_1] \);
(ii) \( \|z(t)\| \leq B(\beta), \quad \forall t \geq 0 \);
(iii) \( \|z(t)\| < \varepsilon, \quad \forall t \geq T(\beta, \varepsilon) \).

This definition is clearly a generalization of global asymptotic stability in the sense of Lyapunov for a system of the form (2.1) that has a unique solution. With the help of the notion of GSS, Kurzweil proved [4] that the Lyapunov’s second theorem remains true under the hypothesis that the function \( f(x) \) is continuous.

Theorem 2.2 (Kurzweil) (pp. 23-24, [4]) Suppose there exists a \( C^0 \) Lyapunov function \( V(x) \), which is positive definite and proper, such that

\[
\frac{\partial V}{\partial x} f(x) < 0, \quad \forall x \in \mathbb{R}^n - \{0\}.
\]

Then, the trivial solution \( x = 0 \) of the system (2.1) is globally strongly stable.

We conclude this section with three lemmas whose proofs are a direct application of Young’s inequality. They will be frequently used throughout this paper.

Lemma 2.3 For \( x, y \in \mathbb{R}, \quad p \geq 1 \) is an integer, the following inequalities hold:

\[
|x + y|^p \leq 2^{p-1}|x|^p + |y|^p, \tag{2.2}
\]

\[
(|x| + |y|)^\frac{p}{\beta} \leq |x|^{\frac{p}{\beta}} + |y|^{\frac{p}{\beta}} \leq 2^{\frac{p-1}{\beta}}(|x| + |y|)^{\frac{p}{\beta}}. \tag{2.3}
\]

As a consequence, when \( p \geq 1 \) is an odd integer,

\[
|x - y|^p \leq 2^{p-1}|x^p - y^p|. \tag{2.4}
\]

Lemma 2.4 For any positive real numbers \( c, d \) and any real-valued function \( \gamma(x, y) > 0 \),

\[
|x|^a|y|^d \leq \frac{c}{c + d} \gamma(x, y)|x|^{a + d} + \frac{d}{c + d} \gamma^{\frac{a}{d}}(x, y)|y|^{a + d}. \tag{2.5}
\]

Lemma 2.5 Let \( a \geq 0, \quad b \geq 0, \) be real numbers and \( p \geq 1, \quad \gamma \geq 1, \) be integers. Then,

\[
a^{p-1}b^\gamma \leq a^p + b^p. \tag{2.6}
\]

3 Explicit Construction of \( C^0 \) GSS Controllers

In this section we present the main result of the paper, which shows that the problem of global strong stabilization (GSS)
of the triangular system (1.1) is solvable by continuous state feedback, without requiring any extra conditions. Moreover, a $C^0$ GSS controller can be explicitly constructed, using a new iterative design algorithm that combines the idea of using homogeneous-based Lyapunov functions [22] with the adding a power integrator technique [15].

**Theorem 3.1** For a chain of odd power integrators perturbed by a $C^1$ lower-triangular vector field (1.1), there exists a constructive continuous controller $u = u(x)$ with $u(0) = 0$, such that the trivial solution $x = 0$ of the closed-loop system is globally strongly stable (GSS).

**Remark 3.2** By assumption, the function $f_i(\cdot)$ in (1.1) is $C^1$ and vanishes at the origin. Then, by the Taylor expansion formula with integration remainder, there exists a smooth non-negative function $\gamma_i(x_1, \ldots, x_i)$ such that

$$[f_i(x_1, \ldots, x_i)] \leq \prod_{j=1}^n |x_j| \gamma_i(\cdot), \quad i = 1, \ldots, n.$$  

This property will be used in the proof of Theorem 3.1.

**Proof of Theorem 3.1.** The proof is based on a continuous type of adding a power integrator technique which simultaneously constructs a $C^1$ control Lyapunov function, which is positive definite and proper, as well as a continuous GSS feedback control law.

**Step 1.** Choose the $C^1$ Lyapunov function

$$V_1(x_1) = \frac{x_1^2}{1 + p_1^{-1}}.$$  

Using (1.1) and (3.1), we have

$$\dot{V}_1(x_1) = \frac{\xi_{k}}{\xi_{k}^i} \left[ x_{k}^{p_i - x_k^{p_i}} + x_{k}^{1} (x_k^{p_i} + f_1(x_1)) \right]$$

$$\leq \frac{\xi_{k}}{\xi_{k}^i} \left[ x_{k}^{p_i - x_k^{p_i}} + x_{k}^{1} x_k^{p_i} + x_1 \right] \gamma_i(x_1).$$

Then, the continuous virtual controller $x_1^i$ defined by

$$x_1^{p_i} = -x_1(n + \gamma_1(x_1)) = -x_1 \beta_1(x_1)$$  

yields

$$\dot{V}_1(x_1) \leq -\frac{\xi_{k}}{\xi_{k}^i} x_1^{1 + \frac{1}{p_1}} + x_1^{1} x_k^{p_i - x_k^{p_i}}.$$  

**Inductive Step.** Suppose at step $k - 1$, there are a $C^1$ Lyapunov function $V_{k-1}(x_1, \ldots, x_{k-1})$, which is positive definite and proper, and a set of $C^0$ virtual controllers $x_1^i, \ldots, x_k^i$, defined by

$$x_1^i = 0 \quad \xi_1 = 1 - x_1 \beta_1(x_1)$$

$$x_2^{p_i} = -x_1 \beta_1(x_1) \quad \xi_2 = x_2^{p_i} - x_2^{p_i}$$

$$\vdots$$

$$x_{k-1}^{p_i} \cdots p_i = -x_{k-1} \beta_{k-1}(\cdot) \quad \xi_{k-1} = x_{k-1}^{p_i} \cdots p_i - x_{k-1}^{p_i} \cdots p_i$$

(3.4)

with $\beta_{k-1}(x_1) > 0, \cdots, \beta_{k-1}(x_1, \ldots, x_{k-1}) > 0$, being smooth, such that

$$V_{k-1} \leq -x_{k-1} \sum_{i=1}^{k-1} \xi_{i}^{1 + \frac{1}{p_i}} + \xi_{k-1} \left[ x_{k-1}^{p_i} - x_{k-1}^{p_i} \right]$$

(3.5)

with $q_{k-1} := 1 + \frac{1}{p_1} - \frac{1}{p_1^{k-1}}$. We shall show that (3.5) also holds at step $k$. To see how this can be done, consider the Lyapunov function

$$V_k(x_1, \ldots, x_k) = V_{k-1}(x_1, \ldots, x_{k-1}) + W_k(x_1, \ldots, x_k)$$

$$W_k(\cdot) = \int_{x_k^i}^{x_k^i} \left( x_k^{p_i - p_k - 1} - x_k^{p_i - p_k - 1} \right)^{p_i} dx.$$  

(3.6)

where $q_{k} := 1 + \frac{1}{p_1} - \frac{1}{p_1^{k-1}}$. The function $V_k(x_1, \ldots, x_k)$ thus defined has several important properties listed in the following two propositions whose proofs involve a tedious calculation and can be found in [17].

**Proposition 1** $W_k(x_1, \ldots, x_k)$ is $C^1$. Moreover, for $l = 1, \ldots, k - 1$, the following holds

$$\frac{\partial W_k}{\partial x_l} = \frac{\xi_{k}^{l + \frac{1}{p_1}}}{\xi_{k}^i} x_k^{p_i - p_k - 1} \left( x_k^{p_i - p_k - 1} - x_k^{p_i - p_k - 1} \right) x_l^{p_i - p_k - 1}.$$  

(3.7)

To continue the proof and the construction of a $C^0$ GSS controller, we need to introduce additional two propositions whose proofs are also given in [17]. They are quite useful when estimating the last two terms in the inequality (3.7).

**Proposition 2** There is a non-negative smooth function $\gamma_k(x_1, \ldots, x_k)$ such that

$$[f_k(x_1, \ldots, x_k)] \leq \left( \xi_{k}^{l + \frac{1}{p_1}} + \cdots + \xi_{k}^{l + \frac{1}{p_1}} \right) x_k^{p_i - p_k - 1}.$$  

**Proposition 4** There is a non-negative smooth function $C_k(x_1, \ldots, x_k)$ such that

$$\left| \frac{\partial x_k^{p_i - p_k - 1}}{\partial x_1} \right| \leq \left( \xi_{k}^{l + \frac{1}{p_1}} + \cdots + \xi_{k}^{l + \frac{1}{p_1}} \right) C_k(x_1, \ldots, x_k).$$  

(3.8)

Using Proposition 3 and Lemma 2.4, one obtains the following estimate

$$\left| \xi_{k}^{l + \frac{1}{p_1}} f_k(\cdot) \right| \leq \xi_{k}^{l + \frac{1}{p_1}} \xi_{k}^{l + \frac{1}{p_1}} + \cdots + \xi_{k}^{l + \frac{1}{p_1}} + \xi_{k}^{l + \frac{1}{p_1}} \xi_{k}^{l + \frac{1}{p_1}}.$$  

(3.9)
To estimate the last term in (3.7), we observe from Proposition 1 that for \( l = 1, \ldots, k - 1, \)
\[
\left| \frac{\partial W_k}{\partial x_l} \right| & \leq q_k |x_k - x_1|\left[ \frac{\partial x^{p_1 \cdots p_{l-1}}}{\partial x_1} \right] \\
& \leq a_k |x_1|^{p_{l-1}} \left| \frac{\partial x^{p_1 \cdots p_{l-1}}}{\partial x_1} \right|, \quad a_k > 0. \quad (3.10)
\]

Then, combining (3.10) and Proposition 4 yields
\[
\sum_{l=1}^{k-1} \left| \frac{\partial W_k}{\partial x_l} \right| \leq a_k |x_1|^{p_{l-1}} \sum_{l=1}^{k-1} \left| \frac{\partial x^{p_1 \cdots p_{l-1}}}{\partial x_1} \right| |x_1| \\
& \leq a_k |x_1|^{p_{l-1}} (|x_1| + \cdots + |x_k|) \sum_{l=1}^{k-1} C_{a_k} (\cdot) \\
& \leq \frac{1}{4} \sum_{j=1}^{k-1} \left( \xi_j \right)^{\frac{1}{2}} + \xi_k \left( c_k + \hat{p}_k (\cdot) + \hat{p}_k (\cdot) \right) \quad (3.11)
\]
The last inequality follows from Lemma 2.4.

Substituting the estimates (3.9), (3.8) and (3.11) into (3.7), we arrive at
\[
\dot{V}_k \leq - (n - k + 1) \left( \xi_1 \frac{1}{\sqrt{2}} \frac{1}{x_k} + \cdots + \xi_k \frac{1}{\sqrt{2}} \frac{1}{x_k} \right) \\
+ \xi_k x^{p_a}_{k+1} + \xi_k \left( c_k + \hat{p}_k (\cdot) + \hat{p}_k (\cdot) \right) \quad (3.12)
\]
or, equivalently,\( x^{p_a}_{k+1} = - (\xi k \beta_k (x_1, \cdots, x_k) \right)^{\frac{1}{2}} x^{p_a}_{k+1} \quad (3.13)\)
with \( \beta_k (\cdot) := (n - k + 1 + c_k + \hat{p}_k (\cdot) + \hat{p}_k (\cdot))^{p_1 \cdots p_{l-1}} > 0 \)
being smooth, is such that
\[
\dot{V}_k \leq - (n - k + 1) \sum_{j=1}^{k-1} \left( \xi_j \frac{1}{\sqrt{2}} \frac{1}{x_k} + \xi_k x^{p_a}_{k+1} - x^{p_a}_{k+1} \right)
\]
This completes the inductive proof.

Using repeatedly the above inductive argument, we conclude immediately, at the \( n \)-th step, that there exists a **static continuous** state feedback control law
\[
u = x^{*}_{n+1} = - (\xi_n \beta_n (x_1, \cdots, x_n)) \right)^{\frac{1}{2}} x^{p_a}_{n+1} \quad (3.14)
\]
with \( \xi_n \) being defined by (3.4), and a \( C^1 \) positive definite and proper Lyapunov function \( V_n (x_1, \cdots, x_n) \) of the form (3.6), such that
\[
\dot{V}_n (x_1, \cdots, x_n) \leq - \left( \xi_1 \frac{1}{\sqrt{2}} + \cdots + \xi_n \frac{1}{\sqrt{2}} \right). \quad (3.15)
\]
By a direct calculation, it is easy to verify that the function on the right-hand side of (3.15) is **negative definite**. Using Kurzweil’s Theorem (Theorem 2.2), it is concluded from Proposition 2 and (3.15) that the trivial solution \( x = 0 \) of the closed-loop system (1.1)-(3.14) is **globally strongly stable**.

**Remark 3.3** From the constructive proof of Theorem 3.1, it is not difficult to see that the GSS controller we designed has the following structure
\[
u = x^{p_1 \cdots p_n} = - \beta_n (\cdot) \left( x^{p_1 \cdots p_{n-1}} + \beta_{n-1} (\cdot) \left( x^{p_1 \cdots p_{n-2}} + \cdots + \beta_2 (\cdot) \left( x^{p_1} + \beta_1 (x_1) \right) \right) \right) \quad (3.16)
\]
In other words, \( \nu \) is smooth (i.e., \( \nu \in C^\infty (\mathbb{R}^n, \mathbb{R}) \)) although the GSS controller (3.14) is only **continuous** (\( \nu \in C^0 (\mathbb{R}^n, \mathbb{R}) \)). For this reason, we say, with a little abuse of the terminology, that the GSS controller (3.14) is "almost smooth" in \( \mathbb{R}^n \). \( \blacksquare \)

### 4 Examples and Discussions

In this section we demonstrate, by means of two examples, how \( C^0 \) globally strongly stabilizing (GSS) controllers can be explicitly constructed, using the **continuous type** of adding a power integrator technique developed in the previous section. Simulation results are also presented to illustrate the main features of our continuous feedback control schemes.

**Example 4.1** Consider the planar system
\[
\begin{align*}
\dot{x}_1 &= x_0^3 + x_1 e^{x_1} \\
\dot{x}_2 &= u
\end{align*}
\]
whose linearization has the exactly same form as that of the system (1.2), and hence contains an uncontrollable mode associated with a positive eigenvalue. Due to the existence of the severe nonlinearity \( x_1 e^{x_1} \) which grows exponentially, the system (4.1) is **not homogeneous**. It **neither belongs** to the class of triangular systems considered in [22, 15].

Using the idea of **homogeneous approximation** [12, 13, 9, 10], it is straightforward to prove that there is a continuous state feedback control law, making (4.1) locally asymptotically stable. In fact, a homogeneous approximation of the system (4.1) is given by (1.2) for which a homogeneous stabilizing controller was already known; see, for instance, the papers [12, 13, 6]. Using the theorem on robust stability of homogeneous systems [9, 18], it is immediately concluded that the same homogeneous controller locally asymptotically stabilizes the system (4.1). However, global stabilization of the system (4.1) is, to the best of our knowledge, an open problem. It cannot be solved by any existing methods.

Next we shall show how the global stabilization problem can be solved by the proposed continuous feedback control scheme, without requiring homogeneity or growth conditions of the system.

As proceeded in the proof of Theorem 3.1, we first choose the \( C^1 \) Lyapunov function \( V_1 (x_1) = \frac{3}{4} x_1^2 \), which is positive definite and proper. A simple calculation shows that the virtual controller \( x_3^* = - (x_1 (2 + e^{-x_1})) \) renders
\[
\dot{V}_1 (x_1) \leq - 2x_1^2 + x_1 \left( x_3^2 - x_2^2 \right) \quad (4.2)
\]

Denote \( \xi_2 = x_2^2 - x_3^2 \) and construct a positive definite and proper Lyapunov function
\[
V_2 (x_1, x_2) = V_1 (x_1) + \frac{1}{20} \left( x_3^4 - x_2 x_3^2 + \frac{3x_2^4}{4} \right) \quad (4.3)
\]
which is continuously differentiable. Then, the time derivative of $V_2(\cdot)$ along the trajectories of (4.1) is

$$V_2 \leq -2x_1^4 + x_2^2 + \xi_2 \leq \frac{2}{20} - \frac{\alpha(x_1)}{20}(x_2 - x_2^*) + \xi_2 - 2x_1, \quad (4.4)$$

where

$$\alpha(x_1) = 2 + (1 + x_1)e^{x_1}. \quad (4.5)$$

By Lemma 2.4, we have

$$\frac{1}{2} x_1^4 + \frac{3}{4} \xi_2^4 \leq \frac{1}{2} x_1^4 + \frac{3}{4} \xi_2^4 \quad (4.6)$$

Using Lemmas 2.3–2.4, it can be shown that

$$\left| \frac{1}{20}(x_2 - x_2^*)\alpha(x_1) (\xi_2 - 2x_1) \right| \leq \xi_2 \frac{1 + \alpha^2(x_1)}{20} \frac{\alpha(x_1)}{20} + \frac{3x_1^2}{4} + \frac{1}{20} \left[ \frac{\alpha(x_1)}{4} \right]^4 \xi_2^4. \quad (4.7)$$

Substituting (4.6) and (4.7) into (4.4), we have

$$V_2 \leq -x_1^4 + \frac{1}{20} \xi_2 w + \frac{3}{4} \frac{1 + \alpha^2(x_1)}{20} \frac{\alpha(x_1)}{4} \xi_2^4 + \frac{1}{20} \left[ \frac{\alpha(x_1)}{4} \right]^4. \quad (4.8)$$

Clearly, the continuous state feedback law

$$u = -\xi_2 \left( 35 + \left( \frac{\alpha(x_1)}{4} \right)^4 + \frac{1 + \alpha^2(x_1)}{20} \right) \quad (4.9)$$

is such that

$$V_2 \leq -x_1^4 + \xi_2^4 = -x_1^4 - \left[ x_2^2 + (x_1^2 + e^{x_1}) \right]^4.$$ 

This, in turn, implies that the trivial solution of the closed-loop system (4.1)–(4.8)–(4.5) is GSS.

A simulation result is shown in Fig. 1, illustrating effectiveness of the $C^0$ controller (4.8)–(4.5) as well as dynamic performance of the closed-loop system.

**Remark 4.2** As a consequence of Example 4.1, one concludes that the nonlinear system

$$\begin{align*}
x_1 &= x_2 \\
x_2 &= x_3 \\
x_3 &= u
\end{align*} \quad (4.10)$$

is also *globally strongly stabilizable by continuous state feedback*. A $C^0$ GSS controller can be explicitly constructed, using the *continuous type of adding a power integrator technique* developed in the last section. Indeed, the controller for the system (4.9) can be easily derived based on the result from Example 4.1. It is interesting to notice that (4.9) satisfies *neither the hypothesis (A1) nor (A2)* in [15], and cannot be handled by any existing methods. The example (4.9) has clearly illustrated the power of continuous feedback design as well as some significant advantages of continuous feedback over smooth feedback.

The next example illustrates how a $C^0$ GSS controller can be designed in the three-dimensional case.

**Example 4.3** Consider the nonlinear system

$$\begin{align*}
x_1 &= x_2 \\
x_2 &= x_3 + \frac{1}{2}(\ln(1 + x_2^2) + \sin x_2) \\
x_3 &= u
\end{align*} \quad (4.11)$$

which is continuously differentiable. Then, the time derivative of $V_2(\cdot)$ along the trajectories of (4.1) is

$$V_2 \leq -2x_1^4 + x_2^2 + \xi_2 \leq \frac{2}{20} - \frac{\alpha(x_1)}{20}(x_2 - x_2^*) + \xi_2 - 2x_1, \quad (4.4)$$

where

$$\alpha(x_1) = 2 + (1 + x_1)e^{x_1}. \quad (4.5)$$

By Lemma 2.4, we have

$$\frac{1}{2} x_1^4 + \frac{3}{4} \xi_2^4 \leq \frac{1}{2} x_1^4 + \frac{3}{4} \xi_2^4 \quad (4.6)$$

Using Lemmas 2.3–2.4, it can be shown that

$$\left| \frac{1}{20}(x_2 - x_2^*)\alpha(x_1) (\xi_2 - 2x_1) \right| \leq \xi_2 \frac{1 + \alpha^2(x_1)}{20} \frac{\alpha(x_1)}{20} + \frac{3x_1^2}{4} + \frac{1}{20} \left[ \frac{\alpha(x_1)}{4} \right]^4 \xi_2^4. \quad (4.7)$$

Substituting (4.6) and (4.7) into (4.4), we have

$$V_2 \leq -x_1^4 + \frac{1}{20} \xi_2 w + \frac{3}{4} \frac{1 + \alpha^2(x_1)}{20} \frac{\alpha(x_1)}{4} \xi_2^4 + \frac{1}{20} \left[ \frac{\alpha(x_1)}{4} \right]^4. \quad (4.8)$$

Clearly, the continuous state feedback law

$$u = -\xi_2 \left( 35 + \left( \frac{\alpha(x_1)}{4} \right)^4 + \frac{1 + \alpha^2(x_1)}{20} \right) \quad (4.9)$$

is such that

$$V_2 \leq -x_1^4 + \xi_2^4 = -x_1^4 - \left[ x_2^2 + (x_1(2 + e^{x_1})) \right]^4.$$ 

This, in turn, implies that the trivial solution of the closed-loop system (4.1)–(4.8)–(4.5) is GSS.

A simulation result is shown in Fig. 1, illustrating effectiveness of the $C^0$ controller (4.8)–(4.5) as well as dynamic performance of the closed-loop system.

**Remark 4.2** As a consequence of Example 4.1, one concludes that the nonlinear system

$$\begin{align*}
x_1 &= x_2 \\
x_2 &= x_3 \\
x_3 &= u
\end{align*} \quad (4.10)$$

is also *globally strongly stabilizable by continuous state feedback*. A $C^0$ GSS controller can be explicitly constructed, using the *continuous type of adding a power integrator technique* developed in the last section. Indeed, the controller for the system (4.9) can be easily derived based on the result from Example 4.1. It is interesting to notice that (4.9) satisfies *neither the hypothesis (A1) nor (A2)* in [15], and cannot be handled by any existing methods. The example (4.9) has clearly illustrated the power of continuous feedback design as well as some significant advantages of continuous feedback over smooth feedback.

The next example illustrates how a $C^0$ GSS controller can be designed in the three-dimensional case.

**Example 4.3** Consider the nonlinear system

$$\begin{align*}
x_1 &= x_2 \\
x_2 &= x_3 + \frac{1}{2}(\ln(1 + x_2^2) + \sin x_2) \\
x_3 &= u
\end{align*} \quad (4.11)$$

which renders the trivial solution $x = 0$ of the system (4.10) GSS. The conclusion can actually be verified by a straightforward but tedious calculation, using the $C^1$ control Lyapunov function (which is positive definite and proper)

$$V(x_1, x_2, x_3) = 2x_1^2 + \frac{x_1 + x_2}{2} + \frac{1}{16} \int_{x_2^*}^{x_1} (x^2 - x_3^2)^2 + ds, \quad (4.12)$$

where $x_1^* = -6(x_2 + x_1)$.

The response of the closed-loop system (4.10)–(4.11) is shown in Fig. 2, demonstrating that the controller (4.11) globally strongly stabilizes (GSS) the three-dimensional system (4.10), with a satisfactory dynamic performance.

**5 Conclusions**

The main contribution of this paper has been the development of a constructive algorithm that shows how to explicitly design a *continuous, globally strongly stabilizing (GSS)* controller, for a chain of odd power integrator perturbed by a $C^1$ lower-triangular vector field. The systems under consideration in general involve an *uncontrollable unstable* Jacobian linearization which implies the non-existence of smooth stabilizers. By combining the idea of the use of homogeneous-like Lyapunov functions [22] and extending the *adding a power integrator technique* [16, 15] to its continuous counterpart, we developed a new continuous feedback control scheme that allows one to solve the problem of stabilization in the large, for a class of highly nonlinear systems having a triangular structure, *without imposing any growth conditions* (e.g. [16, 22]) or homogeneous property (e.g. [22]) on the systems.

The machinery developed in the paper, namely a *continuous version of adding a power integrator*, is believed to be very useful in studying a number of robust control problems for uncertain lower-triangular systems. For example, the problems of robust stabilization, adaptive regulation and disturbance attenuation can be naturally considered in the continuous framework. These problems are currently under investigation and the results will be reported elsewhere.
References


Fig. 1: State trajectories of the closed-loop system (4.1)-(4.5) with $x_1(0) = 2, x_3(0) = 3$

Fig. 2: State trajectories of the system (4.10)-(4.11) with $x_1(0) = x_2(0) = x_3(0) = 10$. 