Additional remark on controllability Gramians

Let us consider the problem of minimizing the energy ($L^2$ norm) of an input controlling $0$ to $x$. The time $T$ is now arbitrary.

Note that, since an input on time $T$ may be seen as an input on time $T' > T$ (just use $u \equiv 0$ on an initial interval), this optimal cost is decreasing (to be more precise, non-increasing) as $T$ increases.

The optimum for inputs of any given length $T$ is $\omega = L^#x$ and has norm $\sqrt{x^*W_T^{-1}x}$, where

$$W_T = \int_0^T \Phi(T, s)B(s)\Phi(T, s)^* \, ds = \int_0^T e^{(T-s)A}BB^*e^{(T-s)A^*} \, ds.$$ 

Equivalently, substituting $T - s \to s$:

$$W_T = \int_0^T e^{sA}BB^*e^{sA^*} \, ds.$$

By the remark above on decreasing costs, $x^*W_T^{-1}x \leq x^*W_S^{-1}x$ if $T > S$.

If $A$ is stable (all its eigenvalues have negative real part), this converges as $t \to +\infty$ to

$$W_\infty = \int_0^\infty e^{sA}BB^*e^{sA^*} \, ds$$

(which is finite).

[Remark: for unstable $A$ we don’t expect the min to be positive: if there is an eigenvector with eigenvalue with real part positive, first move a bit in the direction of such an eigenvector, and then let $\omega \equiv 0$, so that instability drives the motion; in this way, arbitrarily small $\omega$ is enough to reach at least some states.]

One can see that $W_\infty$ is invertible. This can be proved as follows. Take any $p \neq 0$. Note that $p^*W_Tp$ is nondecreasing, because it is the integral of a nonnegative function. Moreover, for any finite $T$, e.g. $T = 1$, the (symmetric) matrix $W_T$ is invertible (because for time-invariant continuous-time systems, reachability implies reachability in arbitrarily small positive time), and hence positive definite. Thus $p^*W_\infty p \geq p^*W_1 p > 0$ and $W_\infty$ must also be positive definite.

Since (by definition of $\int_0^\infty$) $W_T \to W_\infty$, it follows that $W_T^{-1} \to W_\infty^{-1}$ too, and therefore

$$x^*W_T^{-1}x \to x^*W_\infty^{-1}x$$

for all $x$.

Because $x^*W_T^{-1}x$ decreases, this limit is also the infimum of the values $x^*W_T^{-1}x$. In other words, the least possible cost (not necessarily achievable exactly).

Note also, for $W = W_\infty$:

$$AW + WA^* = \int_0^\infty A e^{sA}BB^*e^{sA^*} + e^{sA}BB^*e^{sA^*} A^* \, ds = \int_0^\infty \frac{d}{ds}(e^{sA}BB^*e^{sA^*}) \, ds = e^{sA}BB^*e^{sA^*}\bigg|_0^\infty = -BB^*$$

(using stability), so

$$AW + WA^* + BB^* = 0.$$

This is a set of linear equations! (One can prove, and we do this later when talking about Lyapunov functions, that the matrix equation $AX +XA^* = -BB^*$ has a unique solution, which must therefore be $X = W.$)