We consider a process \( \{N(t), t \geq 0\} \) that counts the number of “hits” happening in the interval \([0, t]\).ootnote{We use the terminology “hits” instead of “events” in order not to confuse with the probabilistic use of the word “event”.} We write \( P_n(t) = P(N(t) = n) \) for any time \( t \), and any nonnegative integer \( n \), and assume the following properties:

1. \( P_0(0) = 1 \) (i.e., zero probability of a hit occurring in an interval of length 0).
2. Hits occurring in disjoint time intervals are independent.
3. \( P(N(t) - N(s) = n) = P_n(t - s) \) (probability of \( k \) hits on interval \([t, s]\) depends only on \( t - s \), for all \( t > s \).
4. \( P(N(t) > 1) = o(t) \). (Recall that a function \( f(t) \) is said to be ”\( o(t) \)” if \( \lim_{t \to 0} \frac{f(t)}{t} = 0 \).)
5. \( P(N(t) = n) \) is differentiable in \( t \), for each nonnegative integer \( n \).

**Claim:** \( N(t) \) must be a Poisson process. In other words, there must exist a \( \lambda > 0 \) such that

\[
P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}
\]

(1)

for all \( n \). The proof is as follows.

Note \( P_n(0) = 0 \) for every \( n > 0 \), because \( P_n(0) \) is the probability that \( N(0) = n \), and property 1 says \( P(N(0) = 0) = 1 \).

Expand \( P_1(t) = P_1(0) + P_1'(0)t + o(t) = P_1(0)t + o(t) \) into a first-order Taylor series. Now define \( \lambda := P_1'(0) \), so that

\[
P_1(t) = \lambda t + o(t).
\]

Therefore,

\[
P_0(t) = P(N(t) = 0) = 1 - P(N(t) = 1) - P(N(t) > 1) = 1 - \lambda t + o(t),
\]

and thus \( P_0'(0) = -\lambda \). On the other hand:

\[
P_0(t + h) = P(N(t + h) = 0)
\]

\[
= P([N(t) = 0] \text{ and } (N(t + h) - N(t) = 0)]
\]

(using independence)

\[
= P(N(t) = 0)P(N(t + h) - N(t) = 0)
\]

\[
= P(N(t) = 0)P(N(h) = 0) = P_0(t)P_0(h)
\]

so taking derivatives with respect to \( h \) at \( h = 0 \) we obtain: \( P_0'(t) = -\lambda P_0(t) \), and hence, solving the differential equation:

\[
P_0(t) = P_0(0)e^{-\lambda t} = e^{-\lambda t}.
\]

(2)

We will show, by induction on \( n \), that (1) is true. For \( n = 0 \), this is just (2). For any \( n \geq 1 \) and any \( t \geq 0 \),

\[
P_n(t + h) = P(N(t + h) = n) = \sum_{k=0}^{n} P([N(t) = n - k] \text{ and } (N(t + h) - N(t) = k)]
\]

\[
= \sum_{k=0}^{n} P(N(t) = n - k)P(N(t + h) - N(t) = k)
\]

\[
= \sum_{k=0}^{n} P_{n-k}(t)P_k(h)
\]

\[
= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) = P_n(t)e^{-\lambda h} + P_{n-1}(t)(\lambda h + o(h)) + o(h)
\]

so taking derivatives with respect to \( h \) at \( h = 0 \) we obtain: \[ P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \].

We view this as a differential equation on \( P_n(t) \) and use an integrating factor:

\[
\frac{d}{dt}[e^{\lambda t}P_n(t)] = \lambda e^{\lambda t}P_n'(t) + \lambda e^{\lambda t}P_n(t) = \lambda e^{\lambda t}P_{n-1}(t) = \lambda e^{\lambda t}e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}
\]

(induction hypothesis used at the end). So

\[
\frac{d}{dt}[e^{\lambda t}P_n(t)] = \lambda^n \frac{t^{n-1}}{(n-1)!}
\]

and hence integrating:

\[
e^{\lambda t}P_n(t) = \frac{(\lambda t)^n}{n!} + P_n(0) = \frac{(\lambda t)^n}{n!}
\]

(remember that \( P_n(0) = 0 \) for all \( n > 0 \)), so multiplying both sides by \( e^{-\lambda t} \) we obtain the desired formula.

Additional notes on Poisson processes, for Rutgers Math 338, E.D. Sontag