1. Matrix Exponentials:
   (a) defined using power series
   \[ e^{tA} = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots \]
   we may compute \( e^{tA} \) this way, but only for very simple examples
   
   (b) shortcut to compute \( e^{tA} \):  
   Write \( A = S \Lambda S^{-1} \).  
   Then observe that \( e^{tA} = S e^{t\Lambda} S^{-1} \).  
   This works well, for instance, if \( A \) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \) with respective eigenvectors \( v_1, \ldots, v_n \).

In this case, we take \( S = (v_1, \ldots, v_n) \)

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & 0 & 0 & \ddots & 0 \\
0 & \cdots & 0 & 0 & \lambda_n \\
\end{pmatrix}
\]

\[
e^{t\Lambda} = \begin{pmatrix}
    e^{t\lambda_1} & 0 & 0 & \cdots & 0 \\
    0 & e^{t\lambda_2} & 0 & \cdots & 0 \\
    0 & 0 & \ddots & 0 \\
    \vdots & 0 & 0 & \ddots & 0 \\
    0 & \cdots & 0 & 0 & e^{t\lambda_n} \\
\end{pmatrix}
\]

* The point of matrix exponentials is that the solution of \( Y' = AY, Y(0) = Y_0 \) (IVP) is \( Y(t) = e^{tA}Y_0 \), and also provides theoretical understanding.  
(IVP = Initial Value Problem)

* If we only want to compute the solution of IVP, there are shortcuts

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**Important Side Remark:**

The \( S \Lambda S^{-1} \) trick is useful for many other applications than solving differential equations i.e.: Since \( A^k = S \Lambda^k S^{-1} \)  
We can solve difference equations \( Y(k+1) = AY(k) \) with initial condition \( Y(0) = Y_0 \), solution is \( Y(k) = A^kY_0 \)
Solving IVP (3.2)

Case of \( \lambda_1 \neq \lambda_2 \) real eigenvalues.
Find eigenvectors \( v_1, v_2 \) for \( \lambda_1, \lambda_2 \), respectively. General solution of \( Y' = AY \) is \((k_1 e^{\lambda_1 t} v_1) + (k_2 e^{\lambda_2 t} v_2)\) where \( k_1 \) and \( k_2 \) are scalar constants (to be fit to initial conditions)

Case of pair of complex conjugate eigenvalues:
\( \lambda_1, \lambda_2 = a \pm ib \) (\( b \neq 0 \))
write \( e^{\lambda_1 t} v_1 = \) real + imaginary parts (vectors)
\[ = Y_1 + iY_2(t) \]
general solution in "real form" is \( k_1 Y_1(t) + k_2 Y_2(t). \)
[we can ignore \( \lambda_2 \) and \( v_2 \) since they are conjugates of \( \lambda_1 \) and \( v_1 \) and don’t give any new information]

* we call \( \frac{2\pi}{b} \) the natural period of the solution and \( \frac{b}{2\pi} \) the natural frequency.
* \( b \) is related to the oscillation frequency and \( a \) tells you if the oscillation is increasing or decreasing.

\[ e^{\lambda t} = e^{(a+ib)t} = e^{at} e^{ibt} = e^{at} (\cos bt + i \sin bt) = e^{at} (\cos bt + i e^{int} (\sin bt) \]

POSSIBLE PHASE PLANES (3.3 & 3.4)

Case of distinct real eigenvalues \( \lambda_1 \neq \lambda_2, \ v_1, v_2 \) eigenvectors.

1. both \( \lambda_1, \lambda_2 \) are negative: sink
   all trajectories approach the origin tangent to the direction of the eigenvector corresponding to the eigenvalue which is closer to zero.

2. both \( \lambda_1, \lambda_2 \) are positive: source
   all trajectories go away from the origin tangent to the direction of the eigenvector corresponding to the eigenvalue which is closer to zero.

3. \( \lambda_1, \lambda_2 \) have opposite signs: saddle
   - let \( \lambda_1 > 0 \): \( v_1 \) is a line with a positive slope through the origin and the origin is a source.
   - let \( \lambda_2 < 0 \): \( v_2 \) is a line with a negative slope through the origin and the origin is a sink.
4. $\lambda_1, \lambda_2$ are complex: $a \pm ib, \ (b \neq 0)$
   
   (i) $a = 0$ (center)
       - solutions look like ellipses (or circles)
       - to decide if they move clockwise or counterclockwise, just pick one point in the plane and see which direction $Ax$ points to.
       - the plots of $x(t)$ and $y(t)$ vs. time look similar to the graph of cosine.

   (ii) $a < 0$ (spiral sink – stable)
       - trajectories go toward the origin while spiraling around it.
       - the plots of $x(t)$ and $y(t)$ vs. time look similar to the graph of cosine with a damped oscillation

   (iii) $a > 0$ (spiral source – unstable)
       - trajectories go away from the origin while spiraling around it
       - the plots of $x(t)$ and $y(t)$ vs. time look similar to the graph of cosine with an increasing oscillation.